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Theorem 14.29, Jech. For $x_i \in M^{\mathbb{B}}$

$$M[G] \models \varphi(x_1^G, \dots, x_n^G) \Leftrightarrow \llbracket \varphi(x_1, \dots, x_n) \rrbracket^{\mathbb{B}} \in G.$$

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