Ordinals.

Please read LST 75-79. (We do not use the Axiom of Choice in this chapter.)

Definition 1.

(1) *transitive set*
(2) *ordinal*

One may extend the natural numbers to a similar class-size structure, which is suitable for enumerating infinite sets: **On**. Many of the properties of **On** are similar to those of $\mathbb{N}$ and are proved in the same way. We will focus on the differences, and refer to the notes for proofs of the similar properties.

Properties of On that are related to familiar $\mathbb{N}$.

(1) $0 \in \mathbf{On}$. [Proposition 7.1, LST]
(2) **On** is closed under successor. [Proposition 7.2, LST]
(3) If A is a set of ordinals, then $\bigcup A$ is an ordinal. [Proposition 7.3, LST]
(4) If $\alpha \in \beta$ and β is an ordinal, then α is an ordinal. (**On** is a transitive class.) [Proposition 7.4, LST]
(5) **On** is a proper class. [Theorem 7.6, LST]
(6) **On** is totally ordered by $\in$. [Theorem 7.7, LST]
(7) The set of natural numbers equals the set of finite ordinals. [Theorem 2.10, HJ]
(8) If $\alpha, \beta \in \mathbf{On}$, then $\alpha < \beta$ iff $\alpha \subsetneq \beta$. [Proposition 7.9, LST]
(9) If $\alpha, \beta \in \mathbf{On}$ and $\alpha < \beta$, then $S(\alpha) \leq \beta$. [Proposition 7.10, LST]
(10) There do not exist ordinals α, β satisfying $\alpha < \beta < S(\alpha)$. [Proposition 7.11, LST]
(11) If A is a set of ordinals, then $\bigcup A$ is an ordinal that is the least upper bound of A in **On**. [Proposition 7.12, LST]
(12) If A is a nonempty class of ordinals, then $\bigcap A$ is an ordinal that is the greatest lower bound of A in **On**. Moreover, $\bigcap A \in A$, so **On** is "well-ordered in the class sense". [Theorem 7.13, LST]

Definition 2.

(3) *successor ordinal*, *limit ordinal*. (Unlike the notes, we define 0 to be a limit ordinal.)
(4) ω

More properties of ordinals

(13) ω is the first limit ordinal. [Theorem 7.16, LST]
(14) The following are equivalent. [Proposition 7.17, LST]
   (a) α is a limit ordinal.
   (b) For every $\beta < \alpha$ there exists γ satisfying $\beta < \gamma < \alpha$.
   (c) $\alpha = \bigcup \alpha$.
(15) if $\alpha = S(\beta)$, then $\bigcup \alpha = \beta$. [Proposition 7.18, LST]

Practice!

(1) Show that a set X is transitive iff $\bigcup X \subseteq X$ iff $X \subseteq \mathcal{P}(X)$.

(2) How many 4-element sets are transitive?

(3) Give an example of a transitive set that is not an ordinal.