Logic Exercises. (+ some solution sketches!)

(1) Write the following formulas formally in the appropriate language.

(a) (Use the language of equality)
κ_3: "There exist three distinct elements."

$$(\exists x_0)(\exists x_1)(\exists x_2)(\neg(x_0 = x_1) \wedge \neg(x_1 = x_2) \wedge \neg(x_0 = x_2))$$

(b) (Use the language of set theory.)
$\varphi_{\subseteq}(x, y)$: "$x \subseteq y$".

$$(\forall z)((z \in x) \to (z \in y))$$

(c) (Use the language of group theory.)
ε: "Any two multiplicative identity elements are equal".

Suppose that $\varphi_{\mathrm{id}}(x)$ expresses that x is a multiplicative identity element. The desired sentence may written with this abbreviation as

$$(\forall x)(\forall y)((\varphi_{\mathrm{id}}(x) \wedge \varphi_{\mathrm{id}}(y)) \to (x = y))$$

One way to write $\varphi_{\mathrm{id}}(x)$ is as "$(\forall z)((x \cdot z = z) \wedge (z \cdot x = z))$". Subsituting this for the abbreviation yields:

$$(\forall x)(\forall y)(((\forall z)((x \cdot z = z) \wedge (z \cdot x = z)) \wedge (\forall z)((y \cdot z = z) \wedge (z \cdot y = z))) \to (x = y))$$

The above sentence is not in prenex form. In general, a sentence of the form

$$(\forall x)\ ((\forall z)P(x, z)) \to Q(x)),$$

where $Q(x)$ does not depend on z, has prenex form

$$(\forall x)(\exists z)(P(x, z) \to Q(x)).$$

(2) Put the following in prenex form:

$$((\forall x)(x = x) \leftrightarrow (\exists x)(x = x))$$

$$(\exists y)(\exists z)(\forall w)(\forall x)(((y = y) \to (z = z)) \wedge ((w = w) \to (x = x)))$$

(This puts the quantifiers in front. More work is required if you want the quantifier-free part to be written in DNF.)

Additional problems proposed in class:
Give winning strategies for the appropriate quantifiers for each of the following sentences in each of the given structures.

(1) $(\forall x)(\exists y)(x < y)$ in the ordered field $\langle \mathbb{R}; +, -, 0, \cdot, 1, < \rangle$

The appropriate (=winning) quantifier is $\exists$.

Strategy:
– $\forall$ chooses some value for x. ($\exists$ has no control over this choice.)
– $\exists$ chooses $y = x + 1$.

To show that this is a winning strategy, one must argue that $(\forall x)(x < x + 1)$. This is a true statement in any ordered field. To see this, the key points are that (i) $0 < 1$ in any ordered field, (ii) $(a < b) \to (c + a < c + b)$ in any ordered field (so, with (i) we get $(0 < 1) \to (x + 0 < x + 1)$ and then $x + 0 < x + 1$), (iii) $x + 0 = x$ (so with (i) and (ii) we get $x < x + 1$).

(2) $(\exists y)(\forall x)(x < y)$ in the ordered field $\langle \mathbb{R}; +, -, 0, \cdot, 1, < \rangle$

The appropriate (=winning) quantifier is $\forall$.

Strategy:
– $\exists$ chooses some value for y. ($\forall$ has no control over this choice.)
– $\forall$ chooses $x = y$.

(3) $(\forall a)(\exists b)(\forall c)(\exists d)(a^2 + b^2 = c^2 + d^2)$ in the field $\langle \mathbb{R}; +, -, 0, \cdot, 1 \rangle$

The appropriate (=winning) quantifier is $\forall$.

Strategy for $\forall$:
– $\forall$ chooses any $a \in \mathbb{R}$. Since this is meant to be a 'strategy', let's be specific and have $\forall$ choose $a = 0$.
– $\exists$ chooses some b.
– $\forall$ computes $a^2 + b^2$ and chooses c large enough so that $a^2 + b^2 < c^2$. (For specificity, choose $c = |b| + 1$.)
– $\exists$ chooses some d.

Now, in $\mathbb{R}$, squares are nonnegative, so $0 \leq d^2$. Adding c^2 to both sides yields $c^2 \leq c^2 + b^2$. By the choice of c, $a^2 + b^2 < c^2 \leq c^2 + d^2$, so $a^2 + b^2 < c^2 + d^2$. This shows that $\forall$ has won (i.e., following this strategy, $a^2 + b^2 \neq c^2 + d^2$).

(4) $(\forall a)(\exists b)(\forall c)(\exists d)(a^2 + b^2 = c^2 + d^2)$ in the field $\langle \mathbb{C}; +, -, 0, \cdot, 1\rangle$

The appropriate (=winning) quantifier is $\exists$.

Strategy:
- $\forall$ chooses some a.
- $\exists$ chooses $b = 0$.
- $\forall$ chooses some c.
- Now $\exists$ solves the equation $a^2 + b^2 = c^2 + d^2$ for d. That is, $\exists$ lets d be any solution to $x^2 = a^2 + b^2 - c^2$. Every complex number has a complex square root, so there is such a $d = \sqrt{a^2 + b^2 - c^2} \in \mathbb{C}$. $\forall$ chooses such a d and wins.