Forcing. The consistency of $\mathsf{ZFC} + \neg\mathrm{CH}$

Let V be a model of ZFC and let M be a transitive set in V that is countable in V and is a model of ZFC. *Forcing* is a technique to construct an extension $M[G]$ of M in V which satisfies ZFC plus an additional property, e.g., $\neg$CH. The idea is the following. To 'force' $\neg$CH to hold in $M[G]$, we have to arrange that $M[G]$ contains an injection from ω_2 in $M[G]$ to 2^ω.

- M may not contain such an injection $\omega_2 \to 2^\omega$, but it contains finite 'partial descriptions' of such an injection, which can be viewed as finite partial functions $f \subseteq (\omega_2 \times \omega) \times 2$. So, let
$$P = \{f : f \text{ is a function with } \mathrm{dmn}(f) \subseteq \omega_2 \times \omega,\ \mathrm{rng}(f) \subseteq 2,\ |f| < \omega\}.$$
P is partially ordered by $\supseteq$, $P \ni \emptyset$, and $(P, \supseteq, \emptyset)$ is a member of M; P will be called a *forcing order*.
- To be able to assemble a total function $g\colon \omega_2 \times \omega \to 2$ from partial descriptions in such a way that g yields an injection $\omega_2 \to 2^\omega$, we extend M by a set $G \subseteq P$ (note: G may not be a member of M), to get a new set $M^* = M[G]$ such that
  - $M \subseteq M[G]$, $G \in M[G]$ and $M[G]$ is a countable transitive model of ZFC;
  - $g = \bigcup G$ is a function in $M[G]$ which yields an injection $\omega_2 \to 2^\omega$ (for the cardinals ω_2 and ω in $M[G]$).

  A set $G \subseteq P$ used in the construction will be called a *filter P-generic over M*, and the new model $M[G]$ of ZFC will be called a *generic extension of M*.

After we

- define forcing orders P and filters P-generic over M in general, and study their existence and basic properties,

we have to face several major challenges in order to see that the idea sketched above works:

- Describe the members of a generic extension $M[G]$.
- Prove that $M[G]$ is a countable transitive model of ZFC.
- Prove that, under suitable assumptions (satisfied in the example above), the construction $M \mapsto M[G]$ preserves cardinals. In particular, this means that ω, ω_1, and ω_2 are the same cardinals in $M[G]$ as in M; that is, the construction $M \mapsto M[G]$ does not introduce any 'unwanted' bijections $\omega \to \omega_1$ or $\omega_1 \to \omega_2$ in $M[G]$.

1. Forcing Orders and Generic Filters

Definition 1.1. A *forcing order* is a triple $\mathbb{P} = (P, \leq, 1)$ where P is a set, $\leq$ is a reflexive, transitive relation on P, and 1 is the largest element of P, i.e. $p \leq 1$ for all $p \in P$.

Note that antisymmetry is not required for $\leq$ in a forcing order $(P, \leq, 1)$, so $\leq$ may not be a partial order. Nevertheless, it is useful to think of the members of P as partial descriptions of a set we want to add to our model M, where $p \leq q$ means that p is a finer description than q.

Definition 1.2. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order.

- Two elements $p, q \in P$ are *compatible* (intuitively: have a common refinement) if there exists $r \in P$ such that $r \leq p, q$; otherwise, p, q are called *incompatible*, and we write $p \perp q$;
- A subset A of P is called an *'antichain'* in $\mathbb{P}$ if any two distinct members of A are incompatible;
- A subset D of P is said to be *dense* in $\mathbb{P}$ if for every $p \in P$ there exists $d \in D$ such that $d \leq p$.

Definition 1.3. Let $\mathbb{P} = (P, \leq, 1)$ be a forcing order, and let M be a c.t.m. of ZFC.

- A *filter* on $\mathbb{P}$ is a subset G of P such that
  - for all $p, q \in G$ there exists $r \in G$ such that $r \leq p, q$, and
  - G is *up-closed*, that is, for all $g \in G$ and $p \in P$, if $g \leq p$, then $p \in G$.
- If $\mathbb{P} \in M$ and G is a subset of P (G is not necessarily a member of M), we say that *G is $\mathbb{P}$-generic over M*, provided
  - G is a filter on $\mathbb{P}$, and
  - for every dense subset D in $\mathbb{P}$ such that $D \in M$ we have that $G \cap D \neq \emptyset$.

Our first theorem shows that generic filters are usually not in the ground model.

Theorem 1.4. *Let M be a c.t.m. of* ZFC*, let $\mathbb{P} = (P, \leq, 1) \in M$ be a forcing order, and let G be a filter on $\mathbb{P}$ that is $\mathbb{P}$-generic over M. If $\mathbb{P}$ satisfies the condition*

$$(*) \qquad \text{for every } p \in P \text{ there exist } q, r \in P \text{ such that } q, r \leq p \text{ and } q \perp r,$$

then $G \notin M$.

Proof. We chose $P \in M$. If also $G \in M$, we get that V and M agree on the question of what are the elements of G (and the elements of P), and therefore on the elements of $P - G$. The set $P - G$ is a set in M, by comprehension applied to P in M. $P - G$ is dense in $\mathbb{P}$, by condition $(*)$, which contradicts G being $\mathbb{P}$-generic. $\square$

Theorem 1.5. *If M is a c.t.m. of* ZFC *and $\mathbb{P} = (P, \leq, 1) \in M$ is a forcing order, then for each $p \in P$ there exists a filter G on $\mathbb{P}$ such that G is $\mathbb{P}$-generic over M and $p \in G$.*

Proof. Let $\mathcal{D}$ be the set of all dense subsets D of P with $D \in M$. Since M is countable, there exists an onto function $\omega \to \mathcal{D}$, $n \mapsto D_n$. Now define a sequence $\langle q_n \rangle_{n \in \omega}$ of elements of P by recursion as follows: $q_0 = p$ and for all $n \in \omega$, given $q_n \in P$, let q_{n+1} be an element of D_n such that $q_{n+1} \leq q_n$. Then the set

$$G = \{r \in P : q_n \leq r \text{ for some } n \in \omega\}$$

- is a filter on $\mathbb{P}$, and
- is $\mathbb{P}$-generic over M.

Clearly, $p \in G$. $\square$