Equipotence, the Natural Numbers, and Finite Sets

(1) Sets X and Y are equipotent if there exists a bijection $\beta\colon X \to Y$. We write $|X| = |Y|$ to denote this.
(2) $|X| \leq |Y|$ means that there exists an injection $\gamma\colon X \to Y$.
(3) $|X| < |Y|$ means $|X| \leq |Y|$ and $|X| \neq |Y|$.
(4) $\mathbb{N}$ is the intersection of all inductive sets.
(5) A set F is finite if $|F| = |n|$ for some $n \in \mathbb{N}$. (We typically drop the vertical bars on n and write $|F| = n$.) F is infinite iff it is not finite.
(6) For $m, n \in \mathbb{N}$, define $m < n$ if $m \in n$. Define $m \leq n$ if $m < n$ or $m = n$.

Prove the following facts. Here, [HJ] stands for 'Hrbacek and Jech'.

(1) (a) $0 \leq n$ for all $n \in \mathbb{N}$. ([HJ] Lemma 3.2.1.)
(b) $m < S(n)$ iff $m = n$ or $m < n$. ([HJ] Lemma 3.2.1.)
(c) $\langle \mathbb{N}; < \rangle$ is a totally-ordered set. ([HJ] Theorem 3.2.2.) (Show that $<$ is transitive, asymmetric, and that the Law of Trichotomy holds.)

(2) $\langle \mathbb{N}; < \rangle$ is a well-ordered set. ([HJ] Theorem 3.2.4.)

(3) If X is finite and Y is a proper subset of X, then there is no bijection $\beta\colon X \to Y$. ([HJ] Lemma 4.2.2.)

(4) Derive from (3) that if $m \neq n$, then $|m| \neq |n|$. ([HJ] Corollary 4.2.3.)

(5) Show that $\mathbb{N}$ is infinite. ([HJ] Corollary 4.2.3.)

(6) Show that if X is finite and $Y \subseteq X$, then Y is finite and $|Y| \leq |X|$. ([HJ] Theorem 4.2.4.)

(7) Show that if X is finite and $f\colon X \to Z$ is a function, then $f(X)$ is finite. ([HJ] Theorem 4.2.5.)

(8) Show that if X and Y are finite, then $X \cup Y$ is finite. ([HJ] Lemma 4.2.6.)

(9) Show that if X is finite, then $\mathcal{P}(X)$ is finite. ([HJ] Theorem 4.2.8.)

(10) Show that if X is infinite, then $n < |X|$ for all $n \in \mathbb{N}$. ([HJ] Theorem 4.2.9.)

(11) For $m, n \in \mathbb{N}$, $m < n$ holds iff $m \subsetneq n$. ([HJ] Exercise 3.2.7.)

(12) Read and try all exercises in Section 2.2 of [HJ].