Equipotence, the Natural Numbers, and Finite Sets

(1) Sets $X$ and $Y$ are equipotent if there exists a bijection $\beta : X \rightarrow Y$. We write $|X| = |Y|$ to denote this.

(2) $|X| \leq |Y|$ means that there exists an injection $\gamma : X \rightarrow Y$.

(3) $|X| < |Y|$ means $|X| \leq |Y|$ and $|X| \neq |Y|$.

(4) $\mathbb{N}$ is the intersection of all inductive sets.

(5) A set $F$ is finite if $|F| = |n|$ for some $n \in \mathbb{N}$. (We typically drop the vertical bars on $n$ and write $|F| = n$.) $F$ is infinite iff it is not finite.

(6) For $m, n \in \mathbb{N}$, define $m < n$ if $m \in n$. Define $m \leq n$ if $m < n$ or $m = n$.

Prove the following facts. Here, [HJ] stands for ‘Hrbacek and Jech’.

(1) (a) $0 \leq n$ for all $n \in \mathbb{N}$. ([HJ] Lemma 3.2.1.)

(b) $m < S(n)$ iff $m = n$ or $m < n$. ([HJ] Lemma 3.2.1.)

(c) $\langle \mathbb{N}; \prec \rangle$ is a totally-ordered set. ([HJ] Theorem 3.2.2.) (Show that $\prec$ is transitive, asymmetric, and that the Law of Trichotomy holds.)

(2) $\langle \mathbb{N}; \prec \rangle$ is a well-ordered set. ([HJ] Theorem 3.2.4.)

(3) If $X$ is finite and $Y$ is a proper subset of $X$, then there is no bijection $\beta : X \rightarrow Y$. ([HJ] Lemma 4.2.2.)

(4) Derive from (3) that if $m \neq n$, then $|m| \neq |n|$. ([HJ] Corollary 4.2.3.)

(5) Show that $\mathbb{N}$ is infinite. ([HJ] Corollary 4.2.3.)

(6) Show that if $X$ is finite and $Y \subseteq X$, then $Y$ is finite and $|Y| \leq |X|$. ([HJ] Theorem 4.2.4.)

(7) Show that if $X$ is finite and $f : X \rightarrow Z$ is a function, then $f(X)$ is finite. ([HJ] Theorem 4.2.5.)

(8) Show that if $X$ and $Y$ are finite, then $X \cup Y$ is finite. ([HJ] Lemma 4.2.6.)

(9) Show that if $X$ is finite, then $\mathcal{P}(X)$ is finite. ([HJ] Theorem 4.2.8.)

(10) Show that if $X$ is infinite, then $n < |X|$ for all $n \in \mathbb{N}$. ([HJ] Theorem 4.2.9.)

(11) For $m, n \in \mathbb{N}$, $m < n$ holds iff $m \subset n$. ([HJ] Exercise 3.2.7.)

(12) Read and try all exercises in Section 2.2 of [HJ].