Enumeration.

Please read LST 106-110.

The essential fact discussed in today's notes is the ZFC proves that

Every set can be enumerated by an ordinal.

The enumeration of a set can be accomplished by transfinite recursion, and to guarantee that the enumeration process succeeds we invoke the Axiom of Choice. Therefore we start by discussing

The Axiom of Choice (AC). This can be stated in any of the following forms, which are easily seen to be equivalent over ZF. (To fix a choice, we adopt (1) for the statement of AC.)

(1) (Zermelo version of AC) If A is set of pairwise disjoint nonempty sets, then there is a set C such that $|a \cap C| = 1$ for every $a \in A$.
(2) (Wikipedia version) If A is set of pairwise disjoint nonempty sets, then there is a function $c\colon A \to \bigcup A$ satisfying $c(a) \in a$ for every $a \in A$.
(3) (Category theorist's version) Any surjective function $e\colon X \to Y$ has a right inverse $s\colon Y \to X$, $e \circ s = id_Y$. (s is called a *section* of e, and the existence of a section makes e a *split epimorphism*.)
(4) If A is set of nonempty sets, then there is a function $c\colon A \to \bigcup A$ satisfying $c(a) \in a$ for every $a \in A$. (We are not assuming that the members of A are pairwise disjoint.)
(5) If S is a nonempty set, then there is a function $c\colon \mathcal{P}(S)\setminus\{\emptyset\} \to S$ satisfying $c(U) \in U$ for every $U \subseteq S$.

It is less obvious that the following are equivalent over ZF.

(AC) (The Axiom of Choice) Item (1) above.
(WO) (The Well-ordering Principle) Every set can be enumerated by an ordinal.
(ZL) (Zorn's Lemma) Every inductively ordered set has a maximal element.

The proof of the equivalence of these statements requires *transfinite recursion*:

Theorem 1. *(Transfinite Recursion) Let* $\mathbf{G}$ *be a class function. Property* (†) *defines a class function* $\mathbf{F}$ *such that* $\mathbf{F}(\alpha) = \mathbf{G}(\mathbf{F}|_\alpha)$ *for all* $\alpha \in \mathbf{On}$.

Property (†) is the property $\mathbf{P}(x, y)$ that asserts

$$(\dagger) \qquad \begin{cases} x \text{ is an ordinal and } y = t(x) \text{ for some } x\text{-step computation } t \text{ based on } \mathbf{G}, \text{ or} \\ x \text{ is not an ordinal and } y = \emptyset. \end{cases}$$

The proof of the Transfinite Recursion Theorem is like the proof for recursion over $\mathbb{N}$. (It is Theorem 4.5 of [HJ]). The usual proof does not use AC, but it does use most of the other axioms of ZFC. (In particular, it uses the Axiom of Replacement explicitly to form the set of all x-step computations.)

(AC) $\Rightarrow$ (WO)

Assume AC and let S be a set that is to be enumerated by an ordinal. Let $c\colon \mathcal{P}(S)\setminus\{\emptyset\} \to S$ be a choice function for the nonemoty subsets of S. Use transfinite recursion to define an enumeration $\mathbf{E}$ of S by

$$\begin{aligned} \mathbf{E}(0) &= c(S) \\ \mathbf{E}(\alpha) &= c(S \setminus \mathbf{ran}(\mathbf{E}|_\alpha)). \end{aligned}$$

(WO) $\Rightarrow$ (ZL)

Let $\langle P; <\rangle$ be an inductively ordered poset. Let $(p_\alpha)_{\alpha<\gamma}$ be an enumeration of P by the ordinal γ. By transfinite recursion, enumerate a sequence (q_β) by

$$\begin{aligned} q_0 &= p_0 \\ q_\beta &= p_\alpha \text{ for least } \alpha \text{ such that } p_\alpha \text{ strictly dominates all } q_{\beta'},\ \beta' < \beta. \end{aligned}$$

The sequence (q_β) is a strictly ascending chain, which must terminate at a maximal element of $\langle P; <\rangle$.

(ZL) $\Rightarrow$ (AC)

Let S be a nonempty set and let C be the set of all partial choice functions of S. The set $\langle C; \subseteq\rangle$ is inductively ordered, hence has a maximal element. Any maximal partial choice function is a total choice function.

Exercise 1. Another statement equivalent over ZF to AC is

(Basis) Every vector space has a basis.

Give proofs for each of these implications.

(1) (AC) $\Rightarrow$ (Basis)
(2) (WO) $\Rightarrow$ (Basis)
(3) (ZL) $\Rightarrow$ (Basis)

Which of your proofs is the 'easiest' or 'most obvious'?

It is harder to prove that (Basis) $\Rightarrow$ (AC). For this proof, see

Andreas Blass, *Existence of bases implies the Axiom of Choice*, Contemporary Mathematics, Volume 31, 1984, 31–33.

Exercise 2. Prove (WO) $\Rightarrow$ (AC) using this proof idea. Given an enumeration $\mathbf{E}\colon \alpha \to S$ of a nonempty set S, define a choice function $c\colon \mathcal{P}(S)\setminus\{\emptyset\} \to S$ by $c(U) = s \in U$ iff s is the 'earliest' $\mathbf{E}$-enumerated element of U. ($c(U) = \mathbf{E}(\bigcap \mathbf{E}^{-1}(U))$.)

Exercise 3. Let $\mathbf{E}\colon \alpha \to S$ be an enumeration of S by an ordinal α and consider any subset $R \subseteq S$. Show that there is an 'induced' enumeration $\mathbf{E}'\colon \beta \to R$ of R by an ordinal β that has the following property: for $r, r' \in R$, the element r is enumerated before r' by $\mathbf{E}$ iff r is enumerated before r' by $\mathbf{E}$'.