Completion of a forcing order, LST 148-159.

Define the following:

- The topology on a forcing order $\mathbb{P} = (P, \leq, 1)$. (An Alexandrov topology: arbitrary intersections of open sets are open. Equivalently, any $p \in \mathbb{P}$ has a least open neighborhood $N_p = (p]$.)
- Regular open sets. ($\overline{Y}^{\circ} := \mathrm{int}(\overline{Y})$. O is RO $\Leftrightarrow O = \overline{O}^{\circ}$.)
- X is dense below $p \in \mathbb{P}$ if $X \cap N_p$ is topologically dense in N_p.
- The algebra $\mathrm{RO}(\mathbb{P})$.
- The monotone function $e\colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})\colon p \mapsto \overline{N}_p^{\circ}$.
- $\mathbb{P}$ is separative. ($\leq$ is a partial order and $(\forall p)(\forall q)((p \not\leq q) \to (\exists r)((r \leq p) \wedge (r \perp q))))$
- The countable chain condition. (c.c.c. topological space, c.c.c. forcing order.)

Some important facts.

- For any topological space the operator $Y \mapsto \overline{Y}^{\circ}$ is idempotent and monotone. It is a closure operator on the topology. (Theorem 13.18 (iii).)
- For any topological space $\mathbb{T}$, $\mathrm{RO}(\mathbb{T})$ is a complete Boolean algebra. (Theorem 13.19.)
- The image of $e\colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})$ is dense. (Theorem 13.20 (i).)
- $p \perp q$ in $\mathbb{P}$ iff $e(p) \cap e(q) = \emptyset$. (Theorem 13.20 (iii).)
- $e(p) \leq e(q)$ iff N_q is dense below p. (Theorem 13.20 (v).)
- $e\colon \mathbb{P} \to \mathrm{RO}(\mathbb{P})$ is an embedding iff $\mathbb{P}$ is separative. (Theorem 13.20 (iii).)