## Cardinal Arithmetic.

Read LST 125-143 and Chapter 9 of [HJ].

**Definition 1.** Given cardinals $\kappa$ and $\lambda$, choose sets $X$ and $Y$ such that $|X| = \kappa$, $|Y| = \lambda$, and $X \cap Y = \emptyset$. (To be specific, choose $X = \kappa \times \{0\}$ and $Y = \lambda \times \{1\}$.)

(1) $\kappa + \lambda = |X \cup Y|$.
(2) $\kappa \cdot \lambda = |X \times Y|$.
(3) $\kappa^\lambda = |X^Y|$.

We can make a similar definition for the sum or product of an infinite sequence $(\kappa_i)_{i\in I}$ of cardinals: Find pairwise disjoint sets $X_i$ with $|X_i| = \kappa_i$ and define $\sum_{i\in I} \kappa_i = |\bigcup_{i\in I} X_i|$, $\prod_{i\in I} \kappa_i = |\prod_{i\in I} X_i|$,

**Definition 2.** The *continuum function* is the function $\kappa \mapsto 2^\kappa$, which maps cardinals to cardinals.

**Definition 3.** The $\beth$ *numbers* are defined recursively by $\beth_0 = \aleph_0$, $\beth_{S(\alpha)} = 2^{\beth_\alpha}$, and $\beth_\lambda = \bigcup_{\alpha<\lambda} \beth_\alpha$ if $\lambda$ is a limit ordinal.

Now assume that $\kappa$ and $\lambda$ are cardinals.

(1) If $\kappa$ and $\lambda$ are finite, then the cardinal arithmetic operations coincide with the ordinal arithmetic operations.
(2) If $0 < \kappa \leq \lambda$ and $\lambda$ is infinite, then $\kappa + \lambda = \kappa \cdot \lambda = \lambda$.
(3) If $0 < \kappa_i$ for all $i \in I$, then $\sum_{i\in I} \kappa_i = |I| \cdot \sup\{\kappa_i \mid i \in I\}$.
(4) (Kőnig's Lemma) If $\kappa_i < \lambda_i$ for all $i \in I$, then $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$.
(5) (Kőnig's Corollary) If $\kappa$ is infinite, then $\kappa < \kappa^{\mathrm{cf}(\kappa)}$.
(6) (Main Theorem of Cardinal Arithmetic) Call $\kappa$ "$\lambda$-reachable from $\mu$" if $\kappa \leq \mu^\lambda$.
Assume that $\kappa \geq 2$ and $\lambda$ is infinite. The value of $\kappa^\lambda$ may be computed as follows:
   (a) (Case $\kappa \leq \lambda$) $\kappa^\lambda = 2^\lambda$.
   (b) (Case $\kappa$ is infinite and $\lambda$-reachable from $\mu$, some $\mu < \kappa$) $\kappa^\lambda = \mu^\lambda$.
   (c) (Case $\kappa$ is infinite and $\kappa$ is not $\lambda$-reachable from $\mu$ for any $\mu < \kappa$)
      (i) $\kappa^\lambda = \kappa$ if $\lambda < \mathrm{cf}(\kappa)$, while
      (ii) $\kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)}$ if $\lambda \geq \mathrm{cf}(\kappa)$.