

Cardinal Numbers.

Please read LST 121-143 (Chapter 12).

Definition 1. An ordinal $\alpha \in \mathbf{On}$ is an *initial ordinal* if it is not equipotent with a smaller ordinal. An initial ordinal may also be called a *cardinal number* or a *cardinal*.

The set of natural numbers is an initial ordinal. We write ω for this set when we want to treat it as an ordinal (i.e., when the order on this set matters), and we write \aleph_0 for this set when we want to think of it as a cardinal number (i.e., when only the equipotence class of the set matters).

In order to determine whether there are infinite cardinals other than \aleph_0 , we need to know how the equipotence relation restricts to \mathbf{On} . The essential facts are:

- (1) For each $\alpha \in \mathbf{On}$ there exists $\beta \in \mathbf{On}$ such that $|\alpha| < |\beta|$.¹
- (2) Equipotence classes of ordinals are *intervals* in the \in -ordering on \mathbf{On} . That is, the set of all ordinals equipotent to α form an interval $[\beta, \gamma)$ in \mathbf{On} for some ordinals β, γ satisfying $\beta \leq \alpha < \gamma$.²

The two items just listed imply that the class of cardinal numbers is a proper class contained in \mathbf{On} . Because \mathbf{On} is well-ordered, any proper subclass can be enumerated by \mathbf{On} . The \aleph -function enumerates the infinite cardinal numbers. This means that the cardinal numbers are

$$\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \aleph_0, \aleph_1, \aleph_2, \dots$$

That is, the set of finite ordinals equals the set of finite cardinals, and the correspondence $\alpha \mapsto \aleph_\alpha$ is an order-preserving class bijection from \mathbf{On} to the class of infinite cardinal numbers. The fact that the class of cardinal numbers is a subclass of \mathbf{On} , the class of cardinal numbers is well-ordered by \in .

It is common to write α^+ to denote $S(\alpha) = \alpha \cup \{\alpha\}$ when α is an ordinal. For a cardinal κ we write κ^+ to mean $S(\kappa)$ when κ is finite, while if $\kappa = \aleph_\alpha$ we write κ^+ to mean $\aleph_{\alpha+}$. In either case, κ^+ is the cardinal immediately succeeding κ . A cardinal κ is a *successor cardinal* if $\kappa = \lambda^+$ for some cardinal λ and κ is a *(weak) limit cardinal* if it is not $\mathbf{0}$ and not a successor cardinal.

The *cofinality* of an ordered set $\langle P; < \rangle$ is the least cardinality *cofinal* subset. (A subset $C \subseteq P$ is cofinal in $\langle P; < \rangle$ if every element of P is \leq some member of C .) In particular, every cardinal has a cofinality, denoted $\text{cf}(\kappa)$. A cardinal κ is *regular* if $\kappa = \text{cf}(\kappa)$, otherwise it is *singular*.

Exercises.

- (1) Show that $\text{cf}(0) = 0$, and $\text{cf}(n) = 1$ if $n \in \omega \setminus \{0\}$.
- (2) Show that $\text{cf}(\aleph_0) = \aleph_0$, so that \aleph_0 is regular.
- (3) Show that $\text{cf}(\aleph_\alpha) = \text{cf}(\alpha)$.
- (4) Show that $\text{cf}(\aleph_\omega) = \aleph_0$, so that \aleph_ω is singular.

¹This follows from Cantor's Theorem.

²This follows from the Cantor-Schröder-Bernstein Theorem.