Boolean-Valued Models

$\mathbb{B}$-valued sets

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V.

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M.

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_{\alpha}^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists \gamma < \alpha)(\mathrm{dom}(x) \subseteq M_{\gamma}^{\mathbb{B}})\}.$$

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists \gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$,

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists\gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$,

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M^{\mathbb{B}}_{\alpha} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists\gamma < \alpha)(\mathrm{dom}(x) \subseteq M^{\mathbb{B}}_{\gamma})\}.$$

Examples.

- $M^{\mathbb{B}}_0 = \varnothing$, since there is no $\gamma < 0$.

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists \gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$,

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists\gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$,

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_{\alpha}^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists\gamma < \alpha)(\mathrm{dom}(x) \subseteq M_{\gamma}^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$, since the only function x with empty domain is the empty function.

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists\gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$, since the only function x with empty domain is the empty function.
- (First interesting case!)

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_{\alpha}^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists \gamma < \alpha)(\mathrm{dom}(x) \subseteq M_{\gamma}^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$, since the only function x with empty domain is the empty function.
- (First interesting case!)

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists\gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$, since the only function x with empty domain is the empty function.
- (First interesting case!) $M_2^{\mathbb{B}} =$

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists \gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$, since the only function x with empty domain is the empty function.
- (First interesting case!) $M_2^{\mathbb{B}} = \{\{(\varnothing, b)\} \mid b \in \mathbb{B}\}$,

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists\gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$, since the only function x with empty domain is the empty function.
- (First interesting case!) $M_2^{\mathbb{B}} = \{\{(\varnothing, b)\} \mid b \in \mathbb{B}\}$, so $M_2^{\mathbb{B}}$ has one element for each element of $\mathbb{B}$.

$\mathbb{B}$-valued sets

Let M be a countable transitive model of ZFC in V. Let $\mathbb{B}$ be a complete Boolean algebra in M. We define the members of the $\mathbb{B}$-valued model $M^{\mathbb{B}}$ by rank, using recursion.

$$M_\alpha^{\mathbb{B}} = \{x \mid x \text{ is a function to } \mathbb{B} \text{ and } (\exists \gamma < \alpha)(\mathrm{dom}(x) \subseteq M_\gamma^{\mathbb{B}})\}.$$

Examples.

- $M_0^{\mathbb{B}} = \varnothing$, since there is no $\gamma < 0$.
- $M_1^{\mathbb{B}} = \{\varnothing\}$, since the only function x with empty domain is the empty function.
- (First interesting case!) $M_2^{\mathbb{B}} = \{\{(\varnothing, b)\} \mid b \in \mathbb{B}\}$, so $M_2^{\mathbb{B}}$ has one element for each element of $\mathbb{B}$.

$x \in M^{\mathbb{B}}$ iff $(\exists \gamma)(x \in M_\gamma^{\mathbb{B}})$. $M^{\mathbb{B}}$ is the class of $\mathbb{B}$-valued sets in M.

Standard elements

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$.

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$.

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

We recursively define *standard* elements of $M^{\mathbb{B}}$:

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

We recursively define *standard* elements of $M^{\mathbb{B}}$:

Definition.

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

We recursively define *standard* elements of $M^{\mathbb{B}}$:

Definition. For each element $x \in M$, let

$$\hat{x} = \{(\hat{y}, 1) \mid y \in x\}.$$

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

We recursively define *standard* elements of $M^{\mathbb{B}}$:

Definition. For each element $x \in M$, let

$$\hat{x} = \{(\hat{y}, 1) \mid y \in x\}.$$

'Hat' is an injective function $\hat{}: M \to M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

We recursively define *standard* elements of $M^{\mathbb{B}}$:

Definition. For each element $x \in M$, let

$$\hat{x} = \{(\hat{y}, 1) \mid y \in x\}.$$

'Hat' is an injective function $\hat{}: M \to M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

Examples.

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

We recursively define *standard* elements of $M^{\mathbb{B}}$:

Definition. For each element $x \in M$, let

$$\hat{x} = \{(\hat{y}, 1) \mid y \in x\}.$$

'Hat' is an injective function $\hat{}: M \to M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

Examples.

1. $\hat{\emptyset}$ is the empty function.

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

We recursively define *standard* elements of $M^{\mathbb{B}}$:

Definition. For each element $x \in M$, let

$$\hat{x} = \{(\hat{y}, 1) \mid y \in x\}.$$

'Hat' is an injective function $\hat{}: M \to M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

Examples.

1. $\hat{\varnothing}$ is the empty function.

Standard elements

If $\mathbb{B}_1$ is a complete subalgebra of $\mathbb{B}_2$, then $M^{\mathbb{B}_1} \subseteq M^{\mathbb{B}_2}$. That is, any $\mathbb{B}_1$-valued set is a $\mathbb{B}_2$-valued set whose values (hereditarily) lie in $\mathbb{B}_1$. Since $\mathbf{2}$ is a complete subalgebra of $\mathbb{B}$ for any complete Boolean algebra $\mathbb{B}$, we get that $M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

We recursively define *standard* elements of $M^{\mathbb{B}}$:

Definition. For each element $x \in M$, let

$$\hat{x} = \{(\hat{y}, 1) \mid y \in x\}.$$

'Hat' is an injective function $\hat{}: M \to M^{\mathbf{2}} \subseteq M^{\mathbb{B}}$.

Examples.

1. $\hat{\varnothing}$ is the empty function.
2. $\hat{1} = \widehat{\{\varnothing\}}$ is the function defined on hatted elements that is the characteristic function of $\{\hat{\varnothing}\}$.

Our language from $M^{\mathbb{B}}$ will be the first-order language of set theory enriched by a family of constants.

The language of $\mathbb{B}$-valued sets

Our language from $M^{\mathbb{B}}$ will be the first-order language of set theory enriched by a family of constants. We will add one constant for each element of $x \in M^{\mathbb{B}}$ and interpret the constant to be x.

The language of $\mathbb{B}$-valued sets

Our language from $M^{\mathbb{B}}$ will be the first-order language of set theory enriched by a family of constants. We will add one constant for each element of $x \in M^{\mathbb{B}}$ and interpret the constant to be x. (So, every element of $M^{\mathbb{B}}$ is named by a constant.)

Assigning truth values to sentences

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$.

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

1. $[\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

1. $[\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$
2. $[\![\sigma \vee \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \vee [\![\tau]\!]^{\mathbb{B}}$

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

1. $[\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$
2. $[\![\sigma \vee \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \vee [\![\tau]\!]^{\mathbb{B}}$
3. $[\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

1. $[\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$
2. $[\![\sigma \vee \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \vee [\![\tau]\!]^{\mathbb{B}}$
3. $[\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$
4. $[\![(\exists x)\sigma(x)]\!]^{\mathbb{B}} = \bigvee_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

1. $[\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$
2. $[\![\sigma \vee \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \vee [\![\tau]\!]^{\mathbb{B}}$
3. $[\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$
4. $[\![(\exists x)\sigma(x)]\!]^{\mathbb{B}} = \bigvee_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$
5. $[\![(\forall x)\sigma(x)]\!]^{\mathbb{B}} = \bigwedge_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

1. $[\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$
2. $[\![\sigma \vee \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \vee [\![\tau]\!]^{\mathbb{B}}$
3. $[\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$
4. $[\![(\exists x)\sigma(x)]\!]^{\mathbb{B}} = \bigvee_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$
5. $[\![(\forall x)\sigma(x)]\!]^{\mathbb{B}} = \bigwedge_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

1. $[\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$
2. $[\![\sigma \vee \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \vee [\![\tau]\!]^{\mathbb{B}}$
3. $[\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$
4. $[\![(\exists x)\sigma(x)]\!]^{\mathbb{B}} = \bigvee_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$
5. $[\![(\forall x)\sigma(x)]\!]^{\mathbb{B}} = \bigwedge_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$

To get the recursion started, we need to decide the values of $[\![x = y]\!]^{\mathbb{B}}$ and $[\![x \in y]\!]^{\mathbb{B}}$.

Assigning truth values to sentences

To every σ, a (fuzzy) truth value for σ will be an element $[\![\sigma]\!]^{\mathbb{B}} \in \mathbb{B}$. We follow these recursive rules for assigning truth:

1. $[\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$
2. $[\![\sigma \vee \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \vee [\![\tau]\!]^{\mathbb{B}}$
3. $[\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$
4. $[\![(\exists x)\sigma(x)]\!]^{\mathbb{B}} = \bigvee_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$
5. $[\![(\forall x)\sigma(x)]\!]^{\mathbb{B}} = \bigwedge_{a \in M^{\mathbb{B}}} [\![\sigma(a)]\!]^{\mathbb{B}}$

To get the recursion started, we need to decide the values of $[\![x = y]\!]^{\mathbb{B}}$ and $[\![x \in y]\!]^{\mathbb{B}}$. This is more delicate.

$[\![x = y]\!]^{\mathbb{B}}$ and $[\![x \in y]\!]^{\mathbb{B}}$

$[\![x = y]\!]^{\mathbb{B}}$ and $[\![x \in y]\!]^{\mathbb{B}}$

We want the following to be true:

$[\![x = y]\!]^{\mathbb{B}}$ and $[\![x \in y]\!]^{\mathbb{B}}$

We want the following to be true:

$$\begin{aligned}
[\![x = y]\!]^{\mathbb{B}} &= [\![(\forall z)((z \in x \rightarrow z \in y) \wedge (z \in y \rightarrow z \in x))]\!]^{\mathbb{B}} \\
&= \left(\bigwedge_{z \in \mathrm{dom}(x)} x(z) \rightarrow [\![z \in y]\!]^{\mathbb{B}}\right) \wedge \left(\bigwedge_{z \in \mathrm{dom}(y)} y(z) \rightarrow [\![z \in x]\!]^{\mathbb{B}}\right)
\end{aligned}$$

We want the following to be true:

$$\begin{aligned} [\![x = y]\!]^{\mathbb{B}} &= [\![(\forall z)((z \in x \to z \in y) \wedge (z \in y \to z \in x))]\!]^{\mathbb{B}} \\ &= (\textstyle\bigwedge_{z \in \mathrm{dom}(x)} x(z) \to [\![z \in y]\!]^{\mathbb{B}}) \wedge (\textstyle\bigwedge_{z \in \mathrm{dom}(y)} y(z) \to [\![z \in x]\!]^{\mathbb{B}}) \end{aligned}$$

and

$[\![x = y]\!]^{\mathbb{B}}$ and $[\![x \in y]\!]^{\mathbb{B}}$

We want the following to be true:

$$\begin{aligned}
[\![x = y]\!]^{\mathbb{B}} &= [\![(\forall z)((z \in x \to z \in y) \wedge (z \in y \to z \in x))]\!]^{\mathbb{B}} \\
&= (\textstyle\bigwedge_{z \in \mathrm{dom}(x)} x(z) \to [\![z \in y]\!]^{\mathbb{B}}) \wedge (\textstyle\bigwedge_{z \in \mathrm{dom}(y)} y(z) \to [\![z \in x]\!]^{\mathbb{B}})
\end{aligned}$$

and

$$[\![x \in y]\!]^{\mathbb{B}} = [\![(\exists z)((z \in y \wedge z = x)]\!]^{\mathbb{B}} = \bigvee_{z \in \mathrm{dom}(y)} (y(z) \wedge [\![z = x]\!]^{\mathbb{B}})$$

We want the following to be true:

$$\begin{aligned} [\![x = y]\!]^{\mathbb{B}} &= [\![(\forall z)((z \in x \to z \in y) \wedge (z \in y \to z \in x))]\!]^{\mathbb{B}} \\ &= (\textstyle\bigwedge_{z \in \mathrm{dom}(x)} x(z) \to [\![z \in y]\!]^{\mathbb{B}}) \wedge (\textstyle\bigwedge_{z \in \mathrm{dom}(y)} y(z) \to [\![z \in x]\!]^{\mathbb{B}}) \end{aligned}$$

and

$$[\![x \in y]\!]^{\mathbb{B}} = [\![(\exists z)((z \in y \wedge z = x)]\!]^{\mathbb{B}} = \bigvee_{z \in \mathrm{dom}(y)} (y(z) \wedge [\![z = x]\!]^{\mathbb{B}})$$

We take as joint recursive definitions the equality of the LHS and the RHS in each of these.

$[\![x = y]\!]^{\mathbb{B}}$ and $[\![x \in y]\!]^{\mathbb{B}}$

We want the following to be true:

$$\begin{aligned} [\![x = y]\!]^{\mathbb{B}} &= [\![(\forall z)((z \in x \to z \in y) \wedge (z \in y \to z \in x))]\!]^{\mathbb{B}} \\ &= (\textstyle\bigwedge_{z \in \mathrm{dom}(x)} x(z) \to [\![z \in y]\!]^{\mathbb{B}}) \wedge (\textstyle\bigwedge_{z \in \mathrm{dom}(y)} y(z) \to [\![z \in x]\!]^{\mathbb{B}}) \end{aligned}$$

and

$$[\![x \in y]\!]^{\mathbb{B}} = [\![(\exists z)((z \in y \wedge z = x)]\!]^{\mathbb{B}} = \bigvee_{z \in \mathrm{dom}(y)} (y(z) \wedge [\![z = x]\!]^{\mathbb{B}})$$

We take as joint recursive definitions the equality of the LHS and the RHS in each of these.

We write $M^{\mathbb{B}} \models \sigma$ to mean $[\![\sigma]\!]^{\mathbb{B}} = 1$.

$[\![x = y]\!]^{\mathbb{B}}$ and $[\![x \in y]\!]^{\mathbb{B}}$

We want the following to be true:

$$\begin{aligned} [\![x = y]\!]^{\mathbb{B}} &= [\![(\forall z)((z \in x \to z \in y) \wedge (z \in y \to z \in x))]\!]^{\mathbb{B}} \\ &= (\textstyle\bigwedge_{z \in \mathrm{dom}(x)} x(z) \to [\![z \in y]\!]^{\mathbb{B}}) \wedge (\textstyle\bigwedge_{z \in \mathrm{dom}(y)} y(z) \to [\![z \in x]\!]^{\mathbb{B}}) \end{aligned}$$

and

$$[\![x \in y]\!]^{\mathbb{B}} = [\![(\exists z)((z \in y \wedge z = x)]\!]^{\mathbb{B}} = \bigvee_{z \in \mathrm{dom}(y)} (y(z) \wedge [\![z = x]\!]^{\mathbb{B}})$$

We take as joint recursive definitions the equality of the LHS and the RHS in each of these.

We write $M^{\mathbb{B}} \models \sigma$ to mean $[\![\sigma]\!]^{\mathbb{B}} = 1$.

This completes the construction of $M^{\mathbb{B}}$, its language, and the assignment of truth values to sentences.

Properties of standard elements

Properties of standard elements

Theorem.

Theorem.

1. If $\varphi(x_1, \dots, x_n)$ is a formula and $m_1, \dots, m_n \in M$, then

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then

Properties of standard elements

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then

$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
2. If φ has bounded quantification, then

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
2. If φ has bounded quantification, then

Properties of standard elements

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
2. If φ has bounded quantification, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbb{B}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$

Properties of standard elements

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
2. If φ has bounded quantification, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbb{B}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
3. In particular, when $\varphi(x, y)$ is $x = y$ or $x \in y$ we get

Properties of standard elements

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
2. If φ has bounded quantification, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbb{B}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
3. In particular, when $\varphi(x, y)$ is $x = y$ or $x \in y$ we get

Properties of standard elements

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
2. If φ has bounded quantification, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbb{B}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
3. In particular, when $\varphi(x, y)$ is $x = y$ or $x \in y$ we get
    1. $m_1 = m_2$ in M iff $M^{\mathbb{B}} \models \hat{m}_1 = \hat{m}_2$.

Properties of standard elements

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
2. If φ has bounded quantification, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbb{B}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
3. In particular, when $\varphi(x, y)$ is $x = y$ or $x \in y$ we get
    1. $m_1 = m_2$ in M iff $M^{\mathbb{B}} \models \hat{m}_1 = \hat{m}_2$.

Properties of standard elements

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then

$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$

2. If φ has bounded quantification, then

$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbb{B}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$

3. In particular, when $\varphi(x, y)$ is $x = y$ or $x \in y$ we get
    1. $m_1 = m_2$ in M iff $M^{\mathbb{B}} \models \hat{m}_1 = \hat{m}_2$.
    2. $m_1 \in m_2$ in M iff $M^{\mathbb{B}} \models \hat{m}_1 \in \hat{m}_2$.

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then

$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$

2. If φ has bounded quantification, then

$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbb{B}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$

3. In particular, when $\varphi(x, y)$ is $x = y$ or $x \in y$ we get
    1. $m_1 = m_2$ in M iff $M^{\mathbb{B}} \models \hat{m}_1 = \hat{m}_2$.
    2. $m_1 \in m_2$ in M iff $M^{\mathbb{B}} \models \hat{m}_1 \in \hat{m}_2$.

Theorem.

1. If $\varphi(x_1, \ldots, x_n)$ is a formula and $m_1, \ldots, m_n \in M$, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbf{2}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
2. If φ has bounded quantification, then
$$M \models \varphi(m_1, \ldots, m_n) \leftrightarrow M^{\mathbb{B}} \models \varphi(\hat{m}_1, \ldots, \hat{m}_n).$$
3. In particular, when $\varphi(x, y)$ is $x = y$ or $x \in y$ we get
   1. $m_1 = m_2$ in M iff $M^{\mathbb{B}} \models \hat{m}_1 = \hat{m}_2$.
   2. $m_1 \in m_2$ in M iff $M^{\mathbb{B}} \models \hat{m}_1 \in \hat{m}_2$.

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC.

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.)

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension:

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof:

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$.

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$. Define $v \in M^{\mathbb{B}}$ by $\mathrm{dom}(v) = \mathrm{dom}(u)$

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$. Define $v \in M^{\mathbb{B}}$ by $\mathrm{dom}(v) = \mathrm{dom}(u)$ and for $x \in \mathrm{dom}(v)$

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$. Define $v \in M^{\mathbb{B}}$ by $\operatorname{dom}(v) = \operatorname{dom}(u)$ and for $x \in \operatorname{dom}(v)$

$$v(x) = u(x) \wedge [\![\varphi(x)]\!]^{\mathbb{B}} \;\; (\in \mathbb{B}).$$

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$. Define $v \in M^{\mathbb{B}}$ by $\mathrm{dom}(v) = \mathrm{dom}(u)$ and for $x \in \mathrm{dom}(v)$

$$v(x) = u(x) \wedge [\![\varphi(x)]\!]^{\mathbb{B}} \ \ (\in \mathbb{B}).$$

Expand $[\![(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}}$

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$. Define $v \in M^{\mathbb{B}}$ by $\mathrm{dom}(v) = \mathrm{dom}(u)$ and for $x \in \mathrm{dom}(v)$

$$v(x) = u(x) \wedge [\![\varphi(x)]\!]^{\mathbb{B}} \ \ (\in \mathbb{B}).$$

Expand $[\![(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}}$ (=1?)

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$. Define $v \in M^{\mathbb{B}}$ by $\mathrm{dom}(v) = \mathrm{dom}(u)$ and for $x \in \mathrm{dom}(v)$

$$v(x) = u(x) \wedge [\![\varphi(x)]\!]^{\mathbb{B}} \ \ (\in \mathbb{B}).$$

Expand $[\![(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}}$ (=1?) to

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$. Define $v \in M^{\mathbb{B}}$ by $\mathrm{dom}(v) = \mathrm{dom}(u)$ and for $x \in \mathrm{dom}(v)$

$$v(x) = u(x) \wedge [\![\varphi(x)]\!]^{\mathbb{B}} \ \ (\in \mathbb{B}).$$

Expand $[\![(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}}$ (=1?) to

$$[\![(\forall x \in v)[(x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}} \wedge [\![(\forall x \in u)[\varphi(x) \to (x \in v)]]\!]^{\mathbb{B}}.$$

If M satisfies ZFC, then $M^{\mathbb{B}}$ satisfies ZFC

The conclusion in the title means that $[\![\sigma]\!]^{\mathbb{B}} = 1$ when σ is an axiom of ZFC. ($M^{\mathbb{B}} \models \sigma$.) The full proof can be found in Bell (v. 12, Oxford Logic Guides). Here we sample the proof by establishing Restricted Comprehension: for any formula $\varphi(x)$,

$$(\forall u)(\exists v)(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]$$

Proof: Choose $u \in M^{\mathbb{B}}$. Define $v \in M^{\mathbb{B}}$ by $\mathrm{dom}(v) = \mathrm{dom}(u)$ and for $x \in \mathrm{dom}(v)$

$$v(x) = u(x) \wedge [\![\varphi(x)]\!]^{\mathbb{B}} \quad (\in \mathbb{B}).$$

Expand $[\![(\forall x)[(x \in v) \leftrightarrow (x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}}$ (=1?) to

$$[\![(\forall x \in v)[(x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}} \wedge [\![(\forall x \in u)[\varphi(x) \rightarrow (x \in v)]]\!]^{\mathbb{B}}.$$

$$(= 1 \wedge 1?)$$

Completion of proof for Restricted Comprehension

Completion of proof for Restricted Comprehension

The first conjunct is 1:

Completion of proof for Restricted Comprehension

The first conjunct is 1:

$$[\![(\forall x \in v)[(x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}}$$

Completion of proof for Restricted Comprehension

The first conjunct is 1:

$$\begin{aligned}&[\![(\forall x \in v)[(x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}}\\ &= \bigwedge_{x \in \mathrm{dom}(v)}([\![x \in u]\!]^{\mathbb{B}} \wedge [\![\varphi(x)]\!]^{\mathbb{B}})\end{aligned}$$

Completion of proof for Restricted Comprehension

The first conjunct is 1:

$$\begin{aligned}
&[\![(\forall x \in v)[(x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}} \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(v)} ([\![x \in u]\!]^{\mathbb{B}} \wedge [\![\varphi(x)]\!]^{\mathbb{B}}) \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(v)} 1 \wedge 1 = 1.
\end{aligned}$$

Completion of proof for Restricted Comprehension

The first conjunct is 1:

$$\begin{aligned}
&[\![(\forall x \in v)[(x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}} \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(v)} ([\![x \in u]\!]^{\mathbb{B}} \wedge [\![\varphi(x)]\!]^{\mathbb{B}}) \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(v)} 1 \wedge 1 = 1.
\end{aligned}$$

The second conjunct is 1:

$$\begin{aligned}
&[\![(\forall x \in u)[\varphi(x) \rightarrow (x \in v)]]\!]^{\mathbb{B}} \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(u)} ([\![\varphi(x)]\!]^{\mathbb{B}} \rightarrow [\![x \in v]\!]^{\mathbb{B}})
\end{aligned}$$

Completion of proof for Restricted Comprehension

The first conjunct is 1:

$$
\begin{aligned}
&[\![(\forall x \in v)[(x \in u) \wedge \varphi(x)]]\!]^{\mathbb{B}} \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(v)}([\![x \in u]\!]^{\mathbb{B}} \wedge [\![\varphi(x)]\!]^{\mathbb{B}}) \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(v)} 1 \wedge 1 = 1.
\end{aligned}
$$

The second conjunct is 1:

$$
\begin{aligned}
&[\![(\forall x \in u)[\varphi(x) \to (x \in v)]]\!]^{\mathbb{B}} \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(u)}([\![\varphi(x)]\!]^{\mathbb{B}} \to [\![x \in v]\!]^{\mathbb{B}}) \\
&= \textstyle\bigwedge_{x \in \mathrm{dom}(u)}(1 \to 1) = 1.
\end{aligned}
$$

The forcing relation $\Vdash$

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "*p forces σ*" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "*p forces σ*" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "*p forces σ*" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "*p forces σ*" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.
6. $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.
6. $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.
6. $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$. $[\![\sigma]\!]^{\mathbb{B}} = 0$ iff $\neg(\exists p)[p \Vdash \sigma]$.

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "*p forces σ*" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.
6. $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$. $\quad [\![\sigma]\!]^{\mathbb{B}} = 0$ iff $\neg(\exists p)[p \Vdash \sigma]$.
7. $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma$ or $q \Vdash \neg\sigma]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "*p forces σ*" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.
6. $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$. $\quad [\![\sigma]\!]^{\mathbb{B}} = 0$ iff $\neg(\exists p)[p \Vdash \sigma]$.
7. $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma$ or $q \Vdash \neg\sigma]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.
6. $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$. $[\![\sigma]\!]^{\mathbb{B}} = 0$ iff $\neg(\exists p)[p \Vdash \sigma]$.
7. $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma$ or $q \Vdash \neg\sigma]$.
8. $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "p *forces* σ" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.
6. $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$. $\quad [\![\sigma]\!]^{\mathbb{B}} = 0$ iff $\neg(\exists p)[p \Vdash \sigma]$.
7. $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma$ or $q \Vdash \neg\sigma]$.
8. $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.

The forcing relation $\Vdash$

Let $\mathbb{P}$ be a forcing poset and let $\mathbb{B}$ be its Boolean completion. For $p \in \mathbb{P}$ and for a sentence σ say that "*p forces σ*" and write "$p \Vdash \sigma$" to mean $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$.

Properties of Forcing.

1. $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
2. $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.
3. $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.
4. $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.
5. For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.
6. $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$. $[\![\sigma]\!]^{\mathbb{B}} = 0$ iff $\neg(\exists p)[p \Vdash \sigma]$.
7. $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma$ or $q \Vdash \neg\sigma]$.
8. $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.
9. If $p \Vdash \sigma$ and $q \leqslant p$, then $q \Vdash \sigma$.

Some proofs

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

$(\Rightarrow)$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$,

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

$(\Rightarrow)$ Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}}$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

$(\Rightarrow)$ Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$,

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

$(\Rightarrow)$ Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$,

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive)

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}}$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$.

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}}$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \not\leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$ iff $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\tau]\!]^{\mathbb{B}}$

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$ iff $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\tau]\!]^{\mathbb{B}}$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$ iff $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\tau]\!]^{\mathbb{B}}$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

- $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$ iff $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\tau]\!]^{\mathbb{B}}$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

- $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$ iff $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\tau]\!]^{\mathbb{B}}$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

- $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$ iff $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\tau]\!]^{\mathbb{B}}$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

- $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.

This follows from the previous two using $\sigma \vee \tau \equiv \neg((\neg(\sigma) \vee (\neg\tau))$.

Some proofs

- $p \Vdash \neg\sigma$ iff $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.

($\Rightarrow$) Assume that $p \Vdash \neg\sigma$. If, for some $q \leqslant p$, we have $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $q \leqslant p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $q = 0$. This is impossible for $q \in \mathbb{P}$, so $\neg(\exists q \leqslant p)[q \Vdash \sigma]$.
($\Leftarrow$, Contrapositive) Assume that $p \Vdash \neg\sigma$ fails to hold. Then $p \nleqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$. Hence $p \wedge [\![\sigma]\!]^{\mathbb{B}} \neq 0$. By the density of $\mathbb{P}$ in $\mathbb{B}$, there exists $q \in \mathbb{P}$ such that $q \leqslant p \wedge [\![\sigma]\!]^{\mathbb{B}}$. We have $q \leqslant p$ and $q \Vdash \sigma$. $\square$

- $p \Vdash \sigma \wedge \tau$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

$p \Vdash \sigma \wedge \tau$ iff $p \leqslant [\![\sigma \wedge \tau]\!]^{\mathbb{B}} = [\![\sigma]\!]^{\mathbb{B}} \wedge [\![\tau]\!]^{\mathbb{B}}$ iff $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\tau]\!]^{\mathbb{B}}$ iff $p \Vdash \sigma$ and $p \Vdash \tau$.

- $p \Vdash \sigma \vee \tau$ iff $(\forall q \leqslant p)(\exists r \leqslant q)[r \Vdash \sigma$ or $r \Vdash \tau]$.

This follows from the previous two using $\sigma \vee \tau \equiv \neg((\neg(\sigma) \vee (\neg\tau))$. $\square$

More proofs

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}}$

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$\begin{aligned}
p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\
& \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\
& \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\
& \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\
& \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})].
\end{aligned}$$

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$\begin{aligned} p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\ & \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})]. \square \end{aligned}$$

- $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma \text{ or } q \Vdash \neg\sigma]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. □

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$\begin{aligned} p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\ & \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})]. \square \end{aligned}$$

- $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma \text{ or } q \Vdash \neg\sigma]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\Box$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$\begin{aligned} p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\ & \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})]. \Box \end{aligned}$$

- $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma$ or $q \Vdash \neg\sigma]$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$\begin{aligned} p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\ & \leftrightarrow p \leqslant \bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \bigwedge_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \bigwedge_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\ & \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})]. \square \end{aligned}$$

- $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma$ or $q \Vdash \neg\sigma]$.

If $p \Vdash \sigma$, take $q = p$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\Box$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$
\begin{aligned}
p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\
& \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\
& \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\
& \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\
& \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})]. \Box
\end{aligned}
$$

- $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma \text{ or } q \Vdash \neg\sigma]$.

If $p \Vdash \sigma$, take $q = p$. Else $p \nleqslant [\![\sigma]\!]^{\mathbb{B}}$,

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\Box$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$\begin{aligned} p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\ & \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})]. \Box \end{aligned}$$

- $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma \text{ or } q \Vdash \neg\sigma]$.

If $p \Vdash \sigma$, take $q = p$. Else $p \nleqslant [\![\sigma]\!]^{\mathbb{B}}$, so $p \wedge ([\![\sigma]\!]^{\mathbb{B}})' \neq 0$;

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$\begin{aligned} p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \textstyle\bigwedge_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\ & \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})]. \square \end{aligned}$$

- $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma \text{ or } q \Vdash \neg\sigma]$.

If $p \Vdash \sigma$, take $q = p$. Else $p \nleqslant [\![\sigma]\!]^{\mathbb{B}}$, so $p \wedge ([\![\sigma]\!]^{\mathbb{B}})' \neq 0$; choose $q \leqslant p \wedge ([\![\sigma]\!]^{\mathbb{B}})'$.

More proofs

- $p \Vdash (\forall x)\varphi(x)$ iff for all $u \in M^{\mathbb{B}}[p \Vdash \varphi(u)]$.

$p \Vdash (\forall x)\varphi(x)$ iff $p \leqslant [\![(\forall x)\varphi(x)]\!]^{\mathbb{B}} = \bigwedge_{u \in M^{\mathbb{B}}} [\![\varphi(u)]\!]^{\mathbb{B}}$. This holds iff $p \Vdash \varphi(u)$ for all $u \in M^{\mathbb{B}}$. $\square$

- For $a \in M$, $p \Vdash (\forall x \in \hat{a})\varphi(x)$ iff $(\forall x \in a)[p \Vdash \varphi(\hat{x})]$.

$$\begin{aligned} p \Vdash (\forall x \in \hat{a})\varphi(x) \quad & \leftrightarrow p \leqslant [\![(\forall x \in \hat{a})\varphi(x)]\!]^{\mathbb{B}} \\ & \leftrightarrow p \leqslant \bigwedge\nolimits_{x \in \mathrm{dom}(\hat{a})} (\hat{a}(x) \to [\![\varphi(x)]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \bigwedge\nolimits_{x \in a} (\hat{a}(\hat{x}) \to [\![\varphi(\hat{x})]\!]^{\mathbb{B}}) \\ & \leftrightarrow p \leqslant \bigwedge\nolimits_{x \in a} [\![\varphi(\hat{x})]\!]^{\mathbb{B}} \\ & \leftrightarrow (\forall x \in a)[p \Vdash \varphi(\hat{x})]. \square \end{aligned}$$

- $(\forall p)(\exists q \leqslant p)[q \Vdash \sigma \text{ or } q \Vdash \neg\sigma]$.

If $p \Vdash \sigma$, take $q = p$. Else $p \nleqslant [\![\sigma]\!]^{\mathbb{B}}$, so $p \wedge ([\![\sigma]\!]^{\mathbb{B}})' \neq 0$; choose $q \leqslant p \wedge ([\![\sigma]\!]^{\mathbb{B}})'$. $\square$

(If time, more proofs!)

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$,

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p,

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$.

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$,

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$,

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$.

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$.

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$,

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$,

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$, so $q \leqslant 0$, a contradiction. Thus $[\![\sigma]\!]^{\mathbb{B}} \neq 1$ implies that there exists q such that $q \nVdash \sigma$. $\Box$

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$, so $q \leqslant 0$, a contradiction. Thus $[\![\sigma]\!]^{\mathbb{B}} \neq 1$ implies that there exists q such that $q \not\Vdash \sigma$. $\square$

- $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$, so $q \leqslant 0$, a contradiction. Thus $[\![\sigma]\!]^{\mathbb{B}} \neq 1$ implies that there exists q such that $q \nVdash \sigma$. $\square$

- $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$, so $q \leqslant 0$, a contradiction. Thus $[\![\sigma]\!]^{\mathbb{B}} \neq 1$ implies that there exists q such that $q \nVdash \sigma$. $\square$

- $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.

Otherwise $p \Vdash \sigma$ and $p \Vdash \neg\sigma]$ hold,

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$, so $q \leqslant 0$, a contradiction. Thus $[\![\sigma]\!]^{\mathbb{B}} \neq 1$ implies that there exists q such that $q \nVdash \sigma$. $\square$

- $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.

Otherwise $p \Vdash \sigma$ and $p \Vdash \neg\sigma]$ hold, so $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}}$

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$, so $q \leqslant 0$, a contradiction. Thus $[\![\sigma]\!]^{\mathbb{B}} \neq 1$ implies that there exists q such that $q \nVdash \sigma$. $\square$

- $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.

Otherwise $p \Vdash \sigma$ and $p \Vdash \neg\sigma]$ hold, so $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$,

(If time, more proofs!)

- $[\![\sigma]\!]^{\mathbb{B}} = 1$ iff $(\forall p)[p \Vdash \sigma]$.

If $[\![\sigma]\!]^{\mathbb{B}} = 1$, then $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ for all p, so $(\forall p)[p \Vdash \sigma]$. If $[\![\sigma]\!]^{\mathbb{B}} \neq 1$, then $([\![\sigma]\!]^{\mathbb{B}})' \neq 0$, so there is a q such that $q \leqslant ([\![\sigma]\!]^{\mathbb{B}})' = [\![\neg\sigma]\!]^{\mathbb{B}}$. This leads to $q \Vdash \neg\sigma$. If we also had $q \Vdash \sigma$, then $q \leqslant [\![\sigma]\!]^{\mathbb{B}}$, so $q \leqslant 0$, a contradiction. Thus $[\![\sigma]\!]^{\mathbb{B}} \neq 1$ implies that there exists q such that $q \nVdash \sigma$. $\square$

- $p \Vdash \sigma$ implies $\neg[p \Vdash \neg\sigma]$.

Otherwise $p \Vdash \sigma$ and $p \Vdash \neg\sigma]$ hold, so $p \leqslant [\![\sigma]\!]^{\mathbb{B}}$ and $p \leqslant [\![\neg\sigma]\!]^{\mathbb{B}} = ([\![\sigma]\!]^{\mathbb{B}})'$, so $p = 0$. This can't happen. $\square$