A comparison of two proofs.

Theorem 1. *Any finitely generated $\mathbb{R}$-vector space has a basis.*

Proof. Assume that V is an $\mathbb{R}$-space that is generated by the finite set $G = \{v_0, v_1, \ldots, v_{n-1}\}$.[1] Examine the vectors in G, in order, and discard those that depend on earlier vectors. Let $\mathcal{B}$ be the set of vectors that remain. $\mathcal{B}$ is a basis for V. $\square$

Theorem 2. *Any $\mathbb{R}$-vector space has a basis.*

Proof. Enumerate the vectors in V with an ordinal α, so that $G = \{v_\beta \mid \beta < \alpha\} = V$. G is a generating set for V. Examine the vectors in G, in order, and discard those that depend on earlier vectors. Let $\mathcal{B}$ be the set of vectors that remain. $\mathcal{B}$ is a basis for V. $\square$

These proofs are missing some 'technical' details, namely

(1) How do we know that $\mathcal{B}$ is a basis? We must show that (i) every vector is generated by $\mathcal{B}$ ($\mathcal{B}$ is a spanning set) and that (ii) no vector in $\mathcal{B}$ is generated by other vectors in $\mathcal{B}$ ($\mathcal{B}$ is an independent set).

(2) What does it mean to 'delete' some vectors from G? It would be better to express the process in a positive way by explaining how to assemble the vectors that are to be 'kept' to create $\mathcal{B}$.

These details are on the other side of this page.

[1]This means that every vector $u \in V$ is expressible as a linear combination of the vectors in G.

Additional details for Proof 1:

We have examined the vectors in the enumerated generating set $G = \{v_0, \ldots, v_{n-1}\}$ and deleted each vector that can be generated by earlier vectors in the enumeration

$$v_0, \cancel{v_1}, \cancel{v_2}, v_3, \ldots, v_{n-3}, \cancel{v_{n-2}}, v_{n-1}$$

to form $\mathcal{B} = \{v_0, v_3, \ldots, v_{n-3}, v_{n-1}\}$. We claimed that $\mathcal{B}$ is an independent generating set. Informally, here is why this claim is true:

(1) The process of deleting a vector v_i from a generating sequence, in the case where $v_i = \sum_{j<i} \alpha_j v_j$ depends on earlier vectors in the generating sequence, cannot alter the set of vectors that can be generated. (Any time you need v_i to generate vectors, use $\sum_{j<i} \alpha_j v_j$ in place of v_i.[2])

(2) Suppose that $\mathcal{B} = \{v_{i_0}, v_{i_2}, \ldots, v_{i_s}\}$, $i_0 < \cdots < i_s$, contains no vector that can be generated by earlier vectors. If $\mathcal{B}$ failed to be independent, then it would satisfy a dependence relation:

$$\alpha_0 v_{i_0} + \cdots + \alpha_s v_{i_s} = 0$$

with not all α_j equal to zero. Let r be the largest subscript where $\alpha_r \neq 0$. Solve for v_{i_r}:

$$v_{i_r} = -\frac{1}{\alpha_r} \sum_{\substack{j=0 \\ j \neq r}}^{s} \alpha_j v_{i_j}.$$

This shows that $v_r \in \mathcal{B}$ is generated by earlier vectors, contrary to the construction of $\mathcal{B}$.

Additional details for Proof 2:

We need the same additional details as in Proof 1, but we also need to know why it is legitimate to 'delete' some vectors from a list and collect the remaining vectors into a set. This is done with transfinite recursion. Namely, define a class function where $\mathbf{F}(\beta)$ is a partial basis created after examining all vectors in G with subscripts $< \beta$. Specifically,

$$\begin{aligned} \mathbf{F}(0) &:= \emptyset \\ \mathbf{F}(S(\gamma)) &:= \mathbf{F}(\gamma) \cup \{v_\gamma\} \text{ if } v_\gamma \notin \langle \mathbf{F}(\gamma) \rangle, \text{ else } \mathbf{F}(S(\gamma)) := \mathbf{F}(\gamma) \\ \mathbf{F}(\lambda) &:= \textstyle\bigcup_{\gamma<\lambda} \mathbf{F}(\gamma) \text{if } \lambda \text{ is a limit ordinal} \end{aligned}$$

Now, starting with $G = \{v_\beta \mid \beta < \alpha\} = V$ and applying transfinite recursion, we may construct $\mathcal{B} = \bigcup \mathbf{ran}(\mathbf{F})$, which is the set of vectors obtained from G by deleting all vectors from G that depend on earlier vectors,

[2] This is informal. A more detailed proof of this statement would use induction.