The Vocabulary of ZFC.

ZFC is a first-order theory whose sole nonlogical symbol (beyond equality) is the binary predicate symbol $\in$. In particular, the following symbols are *not* nonlogical symbols of ZFC:

$$\{,\ \},\ \emptyset,\ \subseteq,\ \cup,\ \cap,\ \bigcup,\ \bigcap,\ \mathcal{P}(X),\ |X|,\ \mathbb{N},\ \omega,\ 0,\ ,1,\ 2,\ 3,\ \ldots$$

The Axioms of Set Theory.

(We need to define the underlined terms to make sense of these axioms.)

Equality

(1) (Extensionality) Two sets are equal if they have the same <u>elements</u>.

Existence of Special Sets

(2) (Empty Set) There is a set with no elements.
(Call it the *empty set* and denote it by $\emptyset, \{\ \}$ or 0.)

(3) (Infinity) There is an <u>inductive</u> set.

Creation of New Sets

(4) (Pairing) If A and B are sets, then there is a set $\{A, B\}$ whose elements are exactly A and B. (We allow the case $A = B$.)

(5) (Union) If A is a set, then there is a set $\bigcup A$ whose elements are exactly the elements of the elements of A.

(6) (Power Set) If A is a set, then there is a set $\mathcal{P}(A)$ whose elements are exactly the <u>subsets</u> of A.

(7) (Separation/Restricted Comprehension) If A is a set and P is a property given by a first-order formula <u>with parameters</u>, then there is a set $\{x \in A \mid P(x)\}$ whose elements are exactly the elements of A that satisfy P.

(8) (Replacement) If A is a set and F is a <u>class function</u> given by a first-order formula with parameters, then there is a set $F(A) = \{F(x) \mid x \in A\}$ whose elements are exactly the <u>F-values</u> of elements of A.

(9) (Choice) If $\{A_i \mid i \in I\}$ is set of nonempty <u>disjoint</u> sets, then there is a set C such that $\underline{|A_i \cap C| = 1}$ for every i.

Sets have Special Properties

(10) (Foundation/Regularity) If A is a nonempty set, then there is a set $B \in A$ such that $A \cap B = \emptyset$.

Remarks.

(1) Zermelo's original list of axioms for set theory, from 1908, contained only seven axioms.
   (a) The Axiom of the Empty Set and the Axiom of Pairing were expressed together as a single axiom called the Axiom of Elementary Sets (Axiom der Elementarmengen).
   (b) Zermelo's original list was missing the Axiom of Replacement. This axiom was suggested later, independently, by Fraenkel and Skolem in 1922.
   (c) Zermelo's original list was missing the Axiom of Foundation. This axiom was suggested later by von Neumann in 1925.

(2) The concept of "first-order formula with parameters" did not exist at the time Zermelo published his list of axioms.

Exercise. Verify that you know all the definitions of the underlined terms.

Exercise. Some functions, relations, and predicates are first-order definable. Write down the formulas that define

(1) $\emptyset$
(2) $\{A\}$, $\{A, B\}$
(3) $\bigcup A$, $A \cup B$,
(4) $\bigcap A$, $A \cap B$
(5) $\mathcal{P}(A)$

Exercise. Write all the axioms as first-order sentences.