Models of Set Theory

We will discuss techniques for constructing models of ZFC, provided ZFC is consistent, methods for checking the axioms, and properties of models expressed by formulas whose 'meaning' does not change in going from one model to a larger model or vice versa.

1. The Set-Theoretical Hierarchy. (LST 161-166)

The hierarchy of sets is defined by recursion as follows.

Theorem 1.1. *There exists a unique class function $V\colon \mathbf{On} \to \mathbf{V}$, $\alpha \mapsto V_\alpha$, satisfying the following conditions:*

- $V_0 = \emptyset$,
- $V_\alpha = \mathcal{P}(V_\beta)$ *if* $\alpha = \beta + 1$, *and*
- $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ *if* α *is a limit ordinal.*

Sketch of Proof. The existence of V can be proved by transfinite recursion. Consider the class function $\mathbf{G}\colon \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ defined as follows:

$$\mathbf{G}(\alpha, x) = \begin{cases} \emptyset & \text{if } x = \emptyset, \\ \mathcal{P}\big(x(\beta)\big) & \text{if } x \text{ is a function with domain } \alpha = \beta + 1, \\ \bigcup_{\beta<\alpha} x(\beta) & \text{if } x \text{ is a function with domain } \alpha \text{ with } \alpha \text{ a limit ordinal,} \\ \emptyset & \text{otherwise.} \end{cases}$$

By the Transfinite Recursion Theorem, there exists a class function $\mathbf{F}\colon \mathbf{On} \to \mathbf{V}$ such that $\mathbf{F}(\alpha) = \mathbf{G}(\alpha, \mathbf{F}\restriction\alpha)$ for all $\alpha \in \mathbf{On}$. It is straightforward to check that $V = \mathbf{F}$ has the required properties.

The uniqueness of V follows easily by transfinite induction. □

Before showing that every set is a member of some V_α ($\alpha \in \mathbf{On}$), recall that a set s is *transitive* if every element of s is a subset of s; i.e., for any sets x, y such that $x \in y \in s$ we have that $x \in s$. The next lemma shows that for every set a there is a smallest transitive set (smallest with respect to $\subseteq$) that contains a as a subset.

Lemma 1.2. *For every set a there exists a unique transitive set b such that*

- $a \subseteq b$, *and*
- $b \subseteq t$ *for all transitive sets t such that $a \subseteq t$.*

Sketch of Proof. Let a be a fixed set. The uniqueness of b satisfying these conditions is clear. To prove the existence of such a b we let $\mathbf{F}\colon \omega \to \mathbf{V}$ be the function obtained by applying the Recursion Theorem to the function $\mathbf{G}\colon \omega \times \mathbf{V} \to \mathbf{V}$ defined by

$$\mathbf{G}(n, x) = \begin{cases} a & \text{if } x = \emptyset,\ n = 0, \\ x(m) \cup \bigcup x(m) & \text{if } x \text{ is a function with domain } n = m + 1 \text{ where } m \in \omega, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $\mathbf{F}(0) = \mathbf{G}(0, \emptyset) = a$ and

$$\mathbf{F}(m+1) = \mathbf{G}(m+1, \mathbf{F}\restriction(m+1)) = \mathbf{F}(m) \cup \bigcup \mathbf{F}(m) \quad \text{for all } m \in \omega.$$

It is not hard to check that $b = \bigcup_{m<\omega} \mathbf{F}(m)$ has the required properties. □

Definition 1.3. For any set a, the unique transitive set b shown to exist in Lemma 1.2 is called *the transitive closure of* a, and is denoted by $\mathrm{trcl}(a)$.

Theorem 1.4. *The set-theoretical hierarchy has the following properties:*

(1) *For every* $\alpha \in \mathbf{On}$,
   (i) V_α *is a transitive set, and*
   (ii) $V_\beta \subseteq V_\alpha$ *for all* $\beta < \alpha$.
(2) *Every set is a member of* V_α *for some* $\alpha \in \mathbf{On}$.
(3) (i) $n \leq |V_n| \in \omega$ *for all* $n \in \omega$, *and*
   (ii) $|V_{\omega+\alpha}| = \beth_\alpha$ *for all* $\alpha \in \mathbf{On}$.

Idea of Proof. (1) Prove (i)–(ii) simultaneously by transfinite induction on α. In the case when $\alpha = \gamma + 1$ is a successor ordinal, show first that $V_\gamma \subseteq V_{\gamma+1}$.

(2) Assume there is a set a such that $a \notin V_\alpha$ for every $\alpha \in \mathbf{On}$. Consider the set

$$A = \{x \in \mathrm{trcl}(a \cup \{a\}) : x \notin V_\alpha \text{ for every } \alpha \in \mathbf{On}\}.$$

Clearly, $a \in A$, so $A \neq \emptyset$. By the Axiom of Foundation there exists $x \in A$ such that $x \cap A = \emptyset$. Show:

- For each $y \in x$ we have $y \notin A$, so there exists $\alpha \in \mathbf{On}$ such that $y \in V_\alpha$; let α_y be the least such ordinal.
- For $\beta = \bigcup_{y \in x} \alpha_y$ we have $x \in V_{\beta+1}$, contradiction.

(3) Proceed by induction on ω for (i), and by transfinite induction for (ii). □

Theorem 1.4(2) allows us to define an important notion of 'rank' for sets.

Definition 1.5. For any set s, the *rank of* s, denoted by $\mathrm{rank}(s)$, is the least ordinal α such that $s \in V_{\alpha+1}$.

The following basic properties of the rank function can be proved, in the given order, by using Definition 1.5, Theorem 1.4(1), and earlier properties. For property (v), apply transfinite induction.

Theorem 1.6. *Let* x *be a set, and let* α *be an ordinal. Then*

(i) $V_\alpha = \{y : \mathrm{rank}(y) < \alpha\}$;

(ii) $\mathrm{rank}(y) < \mathrm{rank}(x)$ *for all* $y \in x$;

(iii) $\mathrm{rank}(y) \leq \mathrm{rank}(x)$ *for all* $y \subseteq x$;

(iv) $\mathrm{rank}(x) = \bigcup_{y \in x}(\mathrm{rank}(y) + 1)$;

(v) $\mathrm{rank}(\alpha) = \alpha$;

(vi) $V_\alpha \cap \mathbf{On} = \alpha$.

2. Models of Set Theory. Checking the Axioms. (LST 160-161, 166-168)

By a *set theory structure* we mean a structure $\mathbf{A} = (A, \in^{\mathbf{A}})$ where A is a set and $\in^{\mathbf{A}} \subseteq A \times A$ is a binary relation on A. It will be useful to allow 'class models' as well, where the 'universe' of the model is a class $\mathbf{M}$, and the interpretation of $\in$ is membership. More formally, this is done by relativizing formulas to $\mathbf{M}$.

Definition 2.1. Let $\mathbf{M}$ be a class of sets defined by a formula $\mu(x)$.

(i) The *relativization of a formula φ to* $\mathbf{M}$ is defined by recursion as follows:
- $(x = y)^{\mathbf{M}}$ is $x = y$;
- $(x \in y)^{\mathbf{M}}$ is $x \in y$;
- $(\neg\varphi)^{\mathbf{M}}$ is $(\neg\varphi^{\mathbf{M}})$;
- $(\varphi \to \psi)^{\mathbf{M}}$ is $(\varphi^{\mathbf{M}} \to \psi^{\mathbf{M}})$;
- $(\forall x\, \varphi)^{\mathbf{M}}$ is $\forall x\, (\mu(x) \to \varphi^{\mathbf{M}})$, which is abbreviated also by $\forall x\, (x \in \mathbf{M} \to \varphi^{\mathbf{M}})$ or by $\forall x \in \mathbf{M}\, \varphi^{\mathbf{M}}$.

(ii) We say that φ *holds in* $\mathbf{M}$ if $\mathsf{ZFC} \vdash \varphi^{\mathbf{M}}$. We may also say *"$\varphi^{\mathbf{M}}$ holds"*.

(iii) For any set Φ of formulas, let $\Phi^{\mathbf{M}} = \{\varphi^{\mathbf{M}} : \varphi \in \Phi\}$.

Theorem 2.2. *Let $\mathbf{M}$ be a class, and let $\Gamma \cup \{\varphi\}$ be a set of sentences in the language of set theory. If $\Gamma \models \varphi$, then $\Gamma^{\mathbf{M}} \models \mathbf{M} \neq \emptyset \to \varphi^{\mathbf{M}}$.*

Sketch of Proof. Let $\mathbf{M}$ and $\Gamma \cup \{\varphi\}$ be as above, and assume $\mathbf{M}$ is defined by the formula $\mu(x)$. Since $\mathbf{M} \neq \emptyset$ is an abbreviation for $\exists x\, \mu(x)$, the theorem asserts

$$(\dagger) \qquad \text{If} \quad \Gamma \models \varphi, \quad \text{then} \quad \Gamma^{\mathbf{M}} \cup \{\exists x\, \mu(x)\} \models \varphi^{\mathbf{M}};$$

Let $\mathbf{A} = (A, E)$ be a set theory structure such that $\mathbf{A} \models \exists x\, \mu(x)$, and define a new structure $\mathbf{B} = (B, F)$ by $B = \{a \in A : \mu(a)\}$ (which may be thought of as $A \cap \mathbf{M}$) and $F = E \cap (B \times B)$. Note that $B \neq \emptyset$, so $\mathbf{B}$ is a set theory structure.

- **Claim.** *For any formula χ and assignment $b\colon \{v_0, v_1, \ldots\} \to B$,*

$$\mathbf{A} \models \chi^{\mathbf{M}}[b] \qquad \Longleftrightarrow \qquad \mathbf{B} \models \chi[b].$$

Proof of Claim. (Induction on the length of χ.) Assume $\chi = \forall v_i\, \varphi$.

$$\begin{aligned}
\mathbf{A} \models (\forall v_i \varphi)^{\mathbf{M}}[b] \;&\Leftrightarrow \mathbf{A} \models (\forall v_i(\mu(v_i) \to \varphi^{\mathbf{M}}))[b] \\
&\Leftrightarrow \mathbf{A} \models (\mu(v_i) \to \varphi^{\mathbf{M}})[b^i_c] \text{ for all } c \in A \\
&\Leftrightarrow \mathbf{A} \models (\mu(v_i) \to \varphi^{\mathbf{M}})[b^i_c] \text{ for all } c \in B \\
&\Leftrightarrow \mathbf{B} \models \varphi[b^i_c] \text{ for all } c \in B \\
&\Leftrightarrow \mathbf{B} \models \forall v_i\, \varphi[b].
\end{aligned}$$

- $(\dagger)$ follows from the Claim:

$$\mathbf{A} \models \Gamma^{\mathbf{M}} \cup \{\exists x \mu(x)\} \;\Rightarrow\; \mathbf{B} \models \Gamma \;\Rightarrow\; \mathbf{B} \models \varphi \;\Rightarrow\; \mathbf{A} \models \varphi^{\mathbf{M}}.$$

$\square$

The following consequence of Theorem 2.2 expresses the basic idea of consistency proofs in set theory. We will see applications of Corollary 2.3 when $\Gamma = \mathsf{ZFC}$ and $\Delta \supset \mathsf{ZFC}$.

Corollary 2.3. *Let Γ and Δ be sets of sentences in the language of set theory. If Γ is consistent and there is a class $\mathbf{M}$ such that*

$$(*) \qquad \Gamma \models \mathbf{M} \neq \emptyset \wedge \varphi^{\mathbf{M}} \quad \text{for all } \varphi \in \Delta,$$

then Δ is consistent.

Sketch of Proof. Assume Δ is inconsistent, i.e. $\Delta \vdash \neg(x = x)$. Then $\Delta \models \neg(x = x)$ by the Soundness Theorem. Hence, by Theorem 2.2, $\Delta^{\mathbf{M}} \models \mathbf{M} \neq \emptyset \to \neg(x = x)$. But then our assumption $(*)$ implies that $\Gamma \models \neg(x = x)$. Hence, by the Completeness Theorem, $\Gamma \vdash \neg(x = x)$, which contradicts the consistency of Γ. □

Notation 2.4. Let $\mathsf{ZF} - \mathsf{Inf}$ denote the set of all axioms in ZF, except the Axiom of Infinity.

The next theorem lists some useful sufficient conditions for the axioms in $\mathsf{ZF} - \mathsf{Inf}$ to hold in a class $\mathbf{M}$. Recall that $x \subseteq y$ is an abbreviation for $\forall z\,(z \in x \to z \in y)$, so its relativization to $\mathbf{M}$ is $\forall z \in \mathbf{M}\,(z \in x \to z \in y)$, that is, $x \cap \mathbf{M} \subseteq y$.

Theorem 2.5. (LST 166-168) *Let $\mathbf{M}$ be a nonempty class.*

- $\mathsf{Ext} :\equiv \forall x\,\forall y\,\big(\forall z\,(z \in x \leftrightarrow z \in y) \to x = y\big)$ *holds in $\mathbf{M}$ if $\mathbf{M}$ is transitive.*
- *For every formula $\varphi = \varphi(z, w_1, \dots, w_n)$ (with free variables among $z, w_1, \dots, w_n$),* $\forall x\,\forall w_1 \dots \forall w_n\,\exists y\,\forall z\,\big(z \in y \leftrightarrow (z \in x \wedge \varphi(z, w_1, \dots, w_n))\big) \in \mathsf{Cmpr}$ *holds in $\mathbf{M}$ if*
$$\mathsf{ZFC} \vdash \forall x, w_1, \dots, w_n \in \mathbf{M}\,\big(\{z \in x : \varphi^{\mathbf{M}}(z, w_1, \dots, w_n)\} \in \mathbf{M}\big).$$
- $\mathsf{Pair} :\equiv \forall x\,\forall y\,\exists z\,(x \in z \wedge y \in z)$ *holds in $\mathbf{M}$ if and only if*
$$\mathsf{ZFC} \vdash \forall x, y \in \mathbf{M}\,\exists z \in \mathbf{M}\,(x \in z \wedge y \in z).$$
- $\mathsf{Uni} :\equiv \forall \mathcal{A}\,\exists B\,\forall x\,\big(\exists A\,(x \in A \wedge A \in \mathcal{A}) \to x \in B\big)$ *holds in $\mathbf{M}$ if*
$$\mathsf{ZFC} \vdash \forall \mathcal{A} \in \mathbf{M}\,\exists B \in \mathbf{M}\,\big(\bigcup \mathcal{A} \subseteq B\big).$$
- *If $\mathbf{M}$ is transitive, then* $\mathsf{Pset} :\equiv \forall A\,\exists Z\,\forall x\,(x \subseteq A \to x \in Z)$ *holds in $\mathbf{M}$ if and only if*
$$\mathsf{ZFC} \vdash \forall A \in \mathbf{M}\,\exists Z \in \mathbf{M}\,\big(\mathcal{P}(A) \cap \mathbf{M} \subseteq Z\big).$$
- *for every formula $\varphi = \varphi(x, y, A, w_1, \dots, w_n)$ (with free variables among $x, y, A, w_1, \dots, w_n$)*
$$\forall A\,\forall w_1 \dots \forall w_n\,\big(\forall x \in A\,\exists! y\,\varphi(x, y, A, w_1, \dots, w_n) \\ \to \exists Y\,\forall x \in A\,\exists y \in Y\,\varphi(x, y, A, w_1, \dots, w_n)\big) \in \mathsf{Repl}$$
holds in $\mathbf{M}$ if $\mathbf{M}$ is transitive and
$$\mathsf{ZFC} \vdash \forall A, w_1, \dots, w_n \in \mathbf{M}\,\big(\forall x \in A\,\exists! y\,\big(y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \dots, w_n)\big) \\ \to \exists Y \in \mathbf{M}\,\big(\{y \in \mathbf{M} : \exists x \in A\,\varphi^{\mathbf{M}}(x, y, A, w_1, \dots, w_n)\} \subseteq Y\big)\big).$$
- $\mathsf{Fnd} :\equiv \forall x\,\big(\exists z\, z \in x \to \exists y\,(y \in x \wedge \neg \exists z\,(z \in x \wedge z \in y))\big)$ *holds in $\mathbf{M}$ if $\mathbf{M}$ is transitive.*

Proof. As an example, we prove the statement on Pset. Let $\mathbf{M}$ be transitive. Then

$$\mathsf{Pset}^{\mathbf{M}} \equiv \forall A \in \mathbf{M}\, \exists Z \in \mathbf{M}\, \forall x \in \mathbf{M}\, (x \cap \mathbf{M} \subseteq A \to x \in Z).$$

Since $\mathbf{M}$ is transitive, $x \in \mathbf{M}$ implies that $x \subseteq \mathbf{M}$. Therefore, in ZFC, we have that

$$\text{for } x \in \mathbf{M}\text{:} \quad x \cap \mathbf{M} \subseteq A \to x \in Z \quad \text{is equivalent to} \quad x \subseteq A \to x \in Z, \text{ and}$$
$$\forall x \in \mathbf{M}\, (x \cap \mathbf{M} \subseteq A \to x \in Z) \quad \text{is equivalent to} \quad \mathcal{P}(A) \cap \mathbf{M} \subseteq Z.$$

□

Corollary 2.6. *V_γ is a model of $\mathsf{ZFC} \setminus \{\mathsf{Repl}\}$ for every limit ordinal $\gamma > \omega$.*

Proof.

- For Ext, Cmpr, Pair, Uni, Pset, Fnd, use Theorem 2.5.
  For Pset: V_γ is transitive. Choose $A \in V_\gamma$. $A \in V_\alpha$ for some $\alpha < \gamma$.
  $A \subseteq V_\alpha \Rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(V_\alpha) = V_{\alpha+1} \Rightarrow \mathcal{P}(A) \in V_{\alpha+2} \subseteq V_\gamma$. Let $Z = \mathcal{P}(A)$.
- $V_\gamma \models \mathsf{AC}^{V_\gamma}$.
  Show that if $A \in V_\gamma$, then a choice set for A in V belongs to V_γ.
- $V_\gamma \models \mathsf{Inf}^{V_\gamma}$, as witnessed by $\omega \in V_\gamma$.

□

Recall that a cardinal κ is called *(strongly) inaccessible* if it is uncountable, regular, and satisfies $2^\lambda < \kappa$ for all cardinals $\lambda < \kappa$.

Corollary 2.7. (See Theorem 14.32, pages 177-179 of LST.)
V_κ is a model of ZFC for every inaccessible cardinal κ.

Proof. To show $V_\kappa \models \mathsf{Repl}$, use Theorem 2.5. Let $A, w_1, \ldots, w_n \in V_\kappa$, and assume that $\forall x \in A\, \exists! y\, \big(y \in V_\kappa \wedge \varphi^{V_\kappa}(x, y, A, w_1, \ldots, w_n)\big)$. For each $x \in A$ let $f(x)$ denote the unique $y \in V_\kappa$ with $\varphi^{V_\kappa}(x, y, A, w_1, \ldots, w_n)$. Now, we have that

- $\kappa = \beth_\kappa$;
- $|A| < \kappa$ and $|f[A]| < \kappa$;
- $\bigcup_{a \in A} \operatorname{rank}(f(a)) < \kappa$, and so $f[A] \subseteq V_\beta$ for some $\beta < \kappa$ (as κ is regular);
- $f[A] \in V_{\beta+1} \subseteq V_\kappa$.

□

3. Absoluteness

Definition 3.1. Let $\mathbf{M}$ and $\mathbf{N}$ be classes such that $\mathbf{M} \subseteq \mathbf{N}$. For a formula $\varphi = \varphi(x_1, \ldots, x_n)$ we say that φ *is absolute for* $\mathbf{M}, \mathbf{N}$ if

$$\mathsf{ZFC} \;\vdash\; \forall x_1, \ldots, x_n \in \mathbf{M} \left(\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \varphi^{\mathbf{N}}(x_1, \ldots, x_n)\right).$$

We say that φ *is absolute for* $\mathbf{M}$ to mean that φ is absolute for $\mathbf{M}, \mathbf{V}$.

More formally, the condition defining absoluteness of φ for $\mathbf{M}, \mathbf{N}$ should be written as

$$\mathsf{ZFC} \;\vdash\; \forall x_1, \ldots, x_n \Bigl(\bigwedge_{1 \le i \le n} \mu(x_i) \to \left(\varphi^{\mu}(x_1, \ldots, x_n) \leftrightarrow \varphi^{\nu}(x_1, \ldots, x_n)\right)\Bigr),$$

where $\mu(y)$ and $\nu(y)$ are the formulas (with possibly other parameters) that define $\mathbf{M}$ and $\mathbf{N}$, respectively, and φ^{μ}, φ^{ν} are the relativizations of φ to $\mathbf{M}$ and $\mathbf{N}$, written without abbreviations (see Definition 2.1).

Definition 3.2. Let $\mathbf{M}$ be a class, and let S be a set of sentences. We will say that $\mathbf{M}$ *is a model of* S if $\mathsf{ZFC} \vdash \mathbf{M} \neq \emptyset$ and $\mathsf{ZFC} \vdash \sigma^{\mathbf{M}}$ for all $\sigma \in S$.

Fact 3.3. *Let S be a set of sentences, and let $\mathbf{M} \subseteq \mathbf{N}$ be classes which are models of S. If*

$$S \models \forall x_1, \ldots, x_n \left(\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)\right),$$

then φ is absolute for $\mathbf{M}, \mathbf{N}$ if and only if ψ is.

Proof. It follows from the assumptions that

- $\mathsf{ZFC} \;\vdash\; \{\mathbf{M}, \mathbf{N} \neq \emptyset\} \cup S^{\mathbf{M}} \cup S^{\mathbf{N}}$, since $\mathbf{M}, \mathbf{N}$ are models of S, and
- $\{\mathbf{M} \neq \emptyset\} \cup S^{\mathbf{M}} \models \forall x_1, \ldots, x_n \in \mathbf{M} \left(\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \psi^{\mathbf{M}}(x_1, \ldots, x_n)\right)$,
  $\{\mathbf{N} \neq \emptyset\} \cup S^{\mathbf{N}} \models \forall x_1, \ldots, x_n \in \mathbf{N} \left(\varphi^{\mathbf{N}}(x_1, \ldots, x_n) \leftrightarrow \psi^{\mathbf{N}}(x_1, \ldots, x_n)\right)$,
  by Theorem 2.2.

Since $\models$ and $\vdash$ are equivalent by the Soundness and Completeness Theorems, we get that

$$\mathsf{ZFC} \;\vdash\; \forall x_1, \ldots, x_n \in \mathbf{M} \left(\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \varphi^{\mathbf{N}}(x_1, \ldots, x_n)\right)$$

if and only if

$$\mathsf{ZFC} \;\vdash\; \forall x_1, \ldots, x_n \in \mathbf{M} \left(\psi^{\mathbf{M}}(x_1, \ldots, x_n) \leftrightarrow \psi^{\mathbf{N}}(x_1, \ldots, x_n)\right).$$ □

We will usually apply Fact 3.3 for the case when $S \subseteq \mathsf{ZFC}$.

3.1. Δ_0-formulas and Absoluteness.

Definition 3.4. The set of Δ_0-*formulas* is the smallest set Γ of formulas satisfying the following conditions:

(i) All atomic formulas are in Γ;
(ii) If $\varphi, \psi \in \Gamma$, then $(\neg\varphi) \in \Gamma$ and $(\varphi \to \psi) \in \Gamma$;
(iii) If $\varphi \in \Gamma$, then the formulas $\forall x\,(x \in y \to \varphi)$ (abbreviated $\forall x \in y\,\varphi$) and $\exists x\,(x \in y \wedge \varphi)$ (abbreviated $\exists x \in y\,\varphi$) are also in Γ.

Lemma 3.5. *If* $\mathbf{M}$ *is a transitive class, then the set* Γ *of all formulas that are absolute for* $\mathbf{M}$ *satisfies conditions* (i)–(iii) *of Definition 3.4.*

Idea of Proof. For example, to show (iii) for $\forall$, it suffices to establish that

$$\forall x, y, \overline{w} \in \mathbf{M}\left(\varphi^{\mathbf{M}}(x,y,\overline{w}) \leftrightarrow \varphi(x,y,\overline{w})\right) \models \\ \forall y, \overline{w} \in \mathbf{M}\left(\forall x \in \mathbf{M}\,(x \in y \to \varphi^{\mathbf{M}}(x,y,\overline{w})) \leftrightarrow \forall x\,(x \in y \to \varphi(x,y,\overline{w}))\right),$$

and then invoke the Completeness Theorem. □

Theorem 3.6. *If* $\mathbf{M}$ *is a transitive class and* φ *is a* Δ_0*-formula, then* φ *is absolute for* $\mathbf{M}$.

Proof. This is an immediate consequence of Definition 3.4 and Lemma 3.5. □

Corollary 3.7. *The following properties of a set* x *can be described by* Δ_0*-formulas, and therefore are absolute for every transitive class:*

(i) *x is an ordinal;*
(ii) *x is a limit ordinal;*
(iii) *x is a successor ordinal;*
(iv) *x is a finite ordinal;*
(v) *x is ω;*
(vi) *x is $n\,(\in \omega)$.*

Proof. E.g. for (i), our definition of 'x is an ordinal' can be described by a Δ_0-formula:

$$\forall y \in x\,\forall z \in y\,(z \in x) \ \wedge\ \forall y \in x\,\forall z \in y\,\forall w \in z\,(w \in y).$$ □

Recall that an n-ary class relation $\mathbf{R}$ (on $\mathbf{V}$) is defined by a formula $\rho(v_0, \dots, v_{n-1})$, and an n-ary class function $\mathbf{F}$ (on $\mathbf{V}$) is defined by a formula $\varphi(v_0, \dots, v_{n-1}, v_n)$ such that, for some set $S\,(\subseteq \mathsf{ZFC})$ of axioms,

(1) $$S \vdash \forall v_0, \dots, v_{n-1}\,\exists! v_n\,\varphi(v_0, \dots, v_{n-1}, v_n).$$

Definition 3.8. With the notation above, let $\mathbf{M}$ be a class such that the sentence in (1) holds in $\mathbf{M}$, i.e.,

$$\mathsf{ZFC} \vdash \forall v_0, \ldots, v_{n-1} \in \mathbf{M}\, \exists! v_n \in \mathbf{M}\, \varphi^{\mathbf{M}}(v_0, \ldots, v_{n-1}, v_n).$$

In this case we will say that $\mathbf{F}$ *is defined in* $\mathbf{M}$, and we define *the relativization of* $\mathbf{F}$ *to* $\mathbf{M}$ as the following class function $\mathbf{F}^{\mathbf{M}}$ on $\mathbf{M}$: for all $x_0, \ldots, x_{n-1} \in \mathbf{M}$,

$$\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \text{the unique } y \in \mathbf{M} \text{ such that } \varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y).$$

If $\mathbf{M}$ and $\mathbf{N}$ are classes such that $\mathbf{M} \subseteq \mathbf{N}$, we say that $\mathbf{F}$ *is absolute for* $\mathbf{M}, \mathbf{N}$ if $\mathbf{F}$ is defined in both $\mathbf{M}$ and $\mathbf{N}$, and the formula φ defining $\mathbf{F}$ is absolute for $\mathbf{M}, \mathbf{N}$.

Fact 3.9. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be classes such that* $\mathbf{M} \subseteq \mathbf{N}$, *and let* $\mathbf{F}$ *be a class function that is defined on both* $\mathbf{M}$ *and* $\mathbf{N}$. *Then the following conditions are equivalent:*

(a) $\mathbf{F}$ *is absolute for* $\mathbf{M}, \mathbf{N}$*;*
(b) *for all* $x_0, \ldots, x_{n-1} \in \mathbf{M}$ *we have that* $\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1})$.

Fact 3.3, Theorem 3.6, and Definition 3.8 imply the following absoluteness results.

Corollary 3.10. *The class relations* (i)–(iii), (xii) *and class functions* (iv)–(xi), (xiii)–(xiv) *below can be defined by formulas that are equivalent, on the basis of* $\mathsf{ZF} - \mathsf{Inf}$, *to* Δ_0*-formulas, and therefore are absolute for all transitive class models of* $\mathsf{ZF} - \mathsf{Inf}$*:*

(i) $x \in y$;
(ii) $x = y$;
(iii) $x \subseteq y$;
(iv) $\{x, y\}$;
(v) $\{x\}$;
(vi) (x, y);
(vii) $\emptyset$;
(viii) $x \cup y$;
(ix) $x \cap y$;
(x) $x \setminus y$;
(xi) $x \cup \{x\}$;
(xii) x *is transitive;*
(xiii) $\bigcup x$;
(xiv) $\bigcap x$ (*with* $\bigcap \emptyset = \emptyset$).

Remark 3.11. For each individual class relation/function in Corollary 3.10 a finite subset S of $\mathsf{ZF} - \mathsf{Inf}$ suffices

- to prove the uniqueness condition (1) for any class function involved, and
- to prove that the defining formula is equivalent to a Δ_0-formula.

Therefore we get the stronger conclusion that the given class relation/function is absolute for all transitive class models of S.

3.2. **Further Absoluteness Results.**

Definition 3.12. Let $\mathbf{M}$ and $\mathbf{N}$ be classes such that $\mathbf{M} \subseteq \mathbf{N}$, and let $\varphi(w_1, \dots, w_n)$ be a formula. We say that φ *is absolute upwards for* $\mathbf{M}, \mathbf{N}$ if

$$\mathsf{ZFC} \vdash \forall x_1, \dots, x_n \in \mathbf{M} \left(\varphi^{\mathbf{M}}(x_1, \dots . x_n) \to \varphi^{\mathbf{N}}(x_1, \dots, x_n)\right),$$

and φ *is absolute downwards for* $\mathbf{M}, \mathbf{N}$ if

$$\mathsf{ZFC} \vdash \forall x_1, \dots, x_n \in \mathbf{M} \left(\varphi^{\mathbf{N}}(x_1, \dots . x_n) \to \varphi^{\mathbf{M}}(x_1, \dots, x_n)\right).$$

Clearly, φ is absolute for $\mathbf{M}, \mathbf{N}$ if and only if φ is both absolute upwards and absolute downwards for $\mathbf{M}, \mathbf{N}$.

Fact 3.13. *If* $\varphi(x_1, \dots, x_n, w_1, \dots, w_m)$ *is absolute for* $\mathbf{M}, \mathbf{N}$*, then*

(i) $\exists x_1 \dots \exists x_n\, \varphi(x_1, \dots, x_n, w_1, \dots, w_m)$ *is absolute upwards for* $\mathbf{M}, \mathbf{N}$*, and*

(ii) $\forall x_1 \dots \forall x_n\, \varphi(x_1, \dots, x_n, w_1, \dots, w_m)$ *is absolute downwards for* $\mathbf{M}, \mathbf{N}$.

The next theorem states that absoluteness is preserved under composition.

Theorem 3.14. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be classes such that* $\mathbf{M} \subseteq \mathbf{N}$*, and suppose that the following are absolute for* $\mathbf{M}, \mathbf{N}$*:*

- *a formula* $\varphi(x_1, \dots, x_n)$*,*
- *an* n*-ary class function* $\mathbf{F}$*,*
- m*-ary class functions* $\mathbf{G}_i$ $(1 \leq i \leq n)$*.*

Then the following are also absolute for $\mathbf{M}, \mathbf{N}$*:*

(i) *the formula* $\varphi\big(\mathbf{G}_1(x_1, \dots, x_m), \dots, \mathbf{G}_n(x_1, \dots, x_m)\big)$*;*

(ii) *the class function* $(x_1, \dots, x_m) \mapsto \mathbf{F}\big(\mathbf{G}_1(x_1, \dots, x_m), \dots, \mathbf{G}_n(x_1, \dots, x_m)\big)$*.*

Idea of Proof. Since $\mathbf{G}_1, \dots, \mathbf{G}_n$ are class functions, there exist two formulas for each one of (i), (ii), one obtained from a formula absolute for $\mathbf{M}, \mathbf{N}$ (by Lemma 3.5) by existential quantification, the other one from a similar formula by universal quantification. For example, for (i), the two formulas are

$$\exists y_1 \dots \exists y_n \left(\bigwedge_{i=1}^{n} \big(y_i = \mathbf{G}_i(x_1, \dots, x_m)\big) \wedge \varphi(y_1, \dots, y_n)\right) \quad \text{and}$$

$$\forall y_1 \dots \forall y_n \left(\bigwedge_{i=1}^{n} \big(y_i = \mathbf{G}_i(x_1, \dots, x_m)\big) \to \varphi(y_1, \dots, y_n)\right).$$

Therefore, the desired absoluteness result follows from Fact 3.13. $\square$

Theorem 3.15. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be classes such that* $\mathbf{M} \subseteq \mathbf{N}$*, and suppose that the following are absolute for* $\mathbf{M}, \mathbf{N}$*:*

- *a formula* $\varphi(y, x_1, \ldots, x_n, w_1, \ldots, w_m)$*,*
- n*-ary class functions* $\mathbf{F}$ *and* $\mathbf{G}$*.*

Then the following are also absolute for $\mathbf{M}, \mathbf{N}$*:*

(i) $z \in \mathbf{F}(x_1, \ldots, x_n)$*;*
(ii) $\mathbf{F}(x_1, \ldots, x_n) \in z$*;*
(iii) $\exists y \in \mathbf{F}(x_1, \ldots, x_n)\, \varphi(y, x_1, \ldots, x_n, w_1, \ldots, w_m)$*;*
(iv) $\forall y \in \mathbf{F}(x_1, \ldots, x_n)\, \varphi(y, x_1, \ldots, x_n, w_1, \ldots, w_m)$*;*
(v) $\mathbf{F}(x_1, \ldots, x_n) = \mathbf{G}(x_1, \ldots, x_n)$*;*
(vi) $\mathbf{F}(x_1, \ldots, x_n) \in \mathbf{G}(x_1, \ldots, x_n)$*.*

Idea of Proof. Use the same trick, combined with Lemma 3.5, as in the preceding proof. □

Corollary 3.16. *The following class relations and class functions are absolute for all transitive class models of* $\mathsf{ZF} - \mathsf{Inf}$*:*

(i) *x is an ordered pair;*
(ii) $A \times B$*;*
(iii) *R is a relation;*
(iv) $\mathrm{dmn}(R)$*;*
(v) $\mathrm{rng}(R)$*;*
(vi) *R is a function;*
(vii) $R(x)$ *(*$\emptyset$ *if R is not a function or* $x \notin \mathrm{dmn}(R)$*);*
(viii) *R is a one-to-one function.*

Idea of Proof. Combine Theorem 3.15 with earlier absoluteness results. For example, (i) can be described by the formula

$$\exists y \in \bigcup x\ \exists z \in \bigcup x\ \big(x = (y, z)\big),$$

where $\bigcup x$ and $x = (y, z)$ are absolute for any transitive model $\mathbf{M}$ of $\mathsf{ZF} - \mathsf{Inf}$, by Corollary 3.10. Hence (i) is absolute for $\mathbf{M}$, by Theorem 3.15. □

3.3. **Applications.**

Theorem 3.17. *If* $\mathbf{M}$ *is a transitive class model of* $\mathsf{ZF}-\mathsf{Inf}$ *and* $\omega \in \mathbf{M}$, *then* Inf *(the Axiom of Infinity) holds in* $\mathbf{M}$.

Proof. $\omega \in \mathbf{M}$ implies that

$$\mathsf{ZFC} \ \vdash \ \exists x \in \mathbf{M} \left(\emptyset \in x \ \wedge \ \forall y \in x \, (y \cup \{y\} \in x)\right).$$

By earlier absoluteness results, the sentence here is equivalent to

$$\mathsf{Inf}^{\mathbf{M}} \equiv \Big(\exists x \left(\emptyset \in x \ \wedge \ \forall y \in x \, (y \cup \{y\} \in x)\right)\Big)^{\mathbf{M}}.$$

Thus we get that $\mathsf{ZFC} \vdash \mathsf{Inf}^{\mathbf{M}}$, that is, Inf holds in $\mathbf{M}$. □

Theorem 3.18. *If* $\mathbf{M}$ *is a transitive class model of* ZF, *then*

- $\emptyset, \omega \in \mathbf{M}$*;*
- $\mathbf{M}$ *is closed under the following set-theoretic operations:*
  - (i) $\cup$,
  - (ii) $\cap$,
  - (iii) $(a,b) \mapsto a \setminus b$,
  - (iv) $(a,b) \mapsto \{a,b\}$,
  - (v) $(a,b) \mapsto (a,b)$,
  - (vi) $x \mapsto x \cup \{x\}$,
  - (vii) $\bigcup$,
  - (viii) $\bigcap$;
- $\omega + 1 \subseteq \mathbf{M}$*; and*
- $[\mathbf{M}]^{<\omega} \subseteq \mathbf{M}$.

3.4. **Absoluteness of Recursive Definitions.**

Theorem 3.19. *Let* $\mathbf{A}$ *be a class, let* $\mathbf{R}$ *be a class relation that is well-founded and set-like on* $\mathbf{A}$*, and let* $\mathbf{G}$ *be a class function* $\mathbf{A} \times \mathbf{V} \to \mathbf{V}$*. By the general Recursion Theorem there exists a unique class function* $\mathbf{F}\colon \mathbf{A} \to \mathbf{V}$ *such that*

$$\mathbf{F}(a) = \mathbf{G}\big(a, \mathbf{F}{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big) \quad \textit{for all } a \in \mathbf{A}.$$

If $\mathbf{M}$ *is a transitive class model of* ZF *such that*

(i) $\mathbf{A}$, $\mathbf{R}$, *and* $\mathbf{G}$ *are absolute for* $\mathbf{M}$,

(ii) $\mathsf{ZFC} \vdash (\mathbf{R}$ *is set-like on* $\mathbf{A})^{\mathbf{M}}$, *and*

(iii) $\mathsf{ZFC} \vdash \forall x \in \mathbf{M} \cap \mathbf{A}\ \big(\mathrm{pred}_{\mathbf{A},\mathbf{R}}(x) \subseteq \mathbf{M}\big)$,

then $\mathbf{F}$ *is absolute for* $\mathbf{M}$.

Proof. By absoluteness, ZFC proves the following:

- $\mathbf{A}^{\mathbf{M}} = \mathbf{A} \cap \mathbf{M}$ and $\mathbf{R}^{\mathbf{M}} = \mathbf{R} \cap (\mathbf{M} \times \mathbf{M})$; so
- every nonempty subset of $\mathbf{A}^{\mathbf{M}}$ has an $\mathbf{R}^{\mathbf{M}}$-minimal element; i.e.,
- $(\mathbf{R}$ is well-founded on $\mathbf{A})^{\mathbf{M}}$.

Now the Recursion Theorem for $\mathbf{A}^{\mathbf{M}}$, $\mathbf{R}^{\mathbf{M}}$, and $\mathbf{G}^{\mathbf{M}}\colon \mathbf{A}^{\mathbf{M}} \times \mathbf{M} \to \mathbf{M}$ yields the existence of a unique class function $\mathbf{H}\colon \mathbf{A}^{\mathbf{M}} \to \mathbf{M}$ such that

$$\mathbf{H}(x) = \mathbf{G}^{\mathbf{M}}\big(x, \mathbf{H}{\restriction}\mathrm{pred}_{\mathbf{A}^{\mathbf{M}},\mathbf{R}^{\mathbf{M}}}(x)\big) \quad \text{for all } x \in \mathbf{A}^{\mathbf{M}}.$$

It follows (cf. Fact 3.9) that $\mathbf{F}$ is absolute for $\mathbf{M}$ if we show that $\mathbf{H}(x) = \mathbf{F}(x)$ for all $x \in \mathbf{A}^{\mathbf{M}}$. Assume $\mathbf{H}(x) \neq \mathbf{F}(x)$ for some $x \in \mathbf{A}^{\mathbf{M}}$, and choose x to be $\mathbf{R}^{\mathbf{M}}$-minimal. Then, by absoluteness and assumption (iii),

$$\mathbf{H}(x) = \mathbf{G}^{\mathbf{M}}\big(x, \mathbf{H}{\restriction}\mathrm{pred}_{\mathbf{A}^{\mathbf{M}},\mathbf{R}^{\mathbf{M}}}(x)\big) = \mathbf{G}\big(x, \mathbf{F}{\restriction}\mathrm{pred}_{\mathbf{A},\mathbf{R}}(x)\big) = \mathbf{F}(x),$$

a contradiction. $\square$

Theorem 3.19. *Let* $\mathbf{A}$ *be a class, let* $\mathbf{R}$ *be a class relation that is well-founded and set-like on* $\mathbf{A}$*, and let* $\mathbf{G}$ *be a class function* $\mathbf{A}\times\mathbf{V}\to\mathbf{V}$*. By the general Recursion Theorem there exists a unique class function* $\mathbf{F}\colon\mathbf{A}\to\mathbf{V}$ *such that*

$$\mathbf{F}(a)=\mathbf{G}\big(a,\mathbf{F}\restriction\mathrm{pred}_{\mathbf{A},\mathbf{R}}(a)\big)\quad \textit{for all } a\in\mathbf{A}.$$

If $\mathbf{M}$ *is a transitive class model of* ZF *such that*

(i) $\mathbf{A}$*,* $\mathbf{R}$*, and* $\mathbf{G}$ *are absolute for* $\mathbf{M}$*,*
(ii) $\mathsf{ZFC}\ \vdash\ (\mathbf{R}$ *is set-like on* $\mathbf{A})^{\mathbf{M}}$*, and*
(iii) $\mathsf{ZFC}\ \vdash\ \forall x\in\mathbf{M}\cap\mathbf{A}\,\big(\mathrm{pred}_{\mathbf{A},\mathbf{R}}(x)\subseteq\mathbf{M}\big)$*,*

then $\mathbf{F}$ *is absolute for* $\mathbf{M}$*.*

Corollary 3.20. *The class functions below are absolute for all transitive class models of* ZF*:*

(i) $\alpha+\beta$*,* $\alpha\cdot\beta$*,* α^β *(ordinal addition, multiplication, and exponentiation);*
(ii) $\mathrm{rank}(x)$*;*
(iii) $\mathrm{trcl}(x)$*.*

Idea of Proof. (ii): $\mathbf{A}=\mathbf{V}$, $\mathbf{R}=\{(x,y):x\in y\}$, and $\mathbf{G}\colon\mathbf{V}\times\mathbf{V}\to\mathbf{V}$ defined by

$$\mathbf{G}(x,f)=\begin{cases}\bigcup_{y\in x}(f(y)\cup\{f(y)\}) & \text{if } f \text{ is a function with domain } x,\\ \emptyset & \text{otherwise.}\end{cases}$$

satisfy the assumptions of Theorem 3.19, and yield the class function $\mathbf{F}=\mathrm{rank}$.

(i), (iii): The classes $\mathbf{A}$, $\mathbf{R}$, $\mathbf{G}$ used earlier for the definition satisfy the assumptions of Theorem 3.19. □

Corollary 3.21. *If* $\mathbf{M}$ *is a transitive class model of* ZF*, then*

$$V_\alpha^{\mathbf{M}}=V_\alpha\cap\mathbf{M}\qquad \textit{for all ordinals } \alpha\in\mathbf{M}.$$

Proof. Let α be an ordinal in $\mathbf{M}$. By Corollary 3.7, α is an ordinal (in $\mathbf{V}$). Since $x\in V_\alpha$ iff $\mathrm{rank}(x)<\alpha$ (see Theorem 1.6(i)), and $\mathrm{rank}(x)$ is absolute for $\mathbf{M}$ (by Corollary 3.20(ii)),

$$x\in V_\alpha^{\mathbf{M}}\quad\text{iff}\quad x\in\mathbf{M}\text{ and }\mathrm{rank}^{\mathbf{M}}(x)<\alpha\quad\text{iff}\quad x\in\mathbf{M}\text{ and }\mathrm{rank}(x)<\alpha\quad\text{iff}\quad x\in V_\alpha\cap\mathbf{M}.$$

□

4. Consistency of 'There Exist No Inaccessible Cardinals'

Recall that a cardinal λ is called *(strongly) inaccessible* if it is uncountable, regular, and satisfies $2^\mu < \lambda$ for all cardinals $\mu < \lambda$.

Theorem 4.1. *If* ZFC *is consistent, then so is* $\mathsf{ZFC}+$[There exist no inaccessible cardinals].

Idea of Proof. Consider the class

$$\mathbf{M} = \{x \in \mathbf{V} : \forall\alpha \left((\alpha \text{ is an inaccessible cardinal}) \to x \in V_\alpha\}\right).$$

Our goal is to show that $\mathbf{M}$ is a model of $\mathsf{ZFC}+$ [There exist no inaccessible cardinals].

Case 1: $\mathbf{M} = \mathbf{V}$. Clearly, $\mathbf{V}$ is a model of ZFC. In this case it is also a model of [There exist no inaccessible cardinals]: Otherwise, if α is an inaccessible cardinal in $\mathbf{V}$, then $\mathbf{V} = \mathbf{M} \subseteq V_\alpha$; this is impossible, since $\alpha \in \mathbf{V} \setminus V_\alpha$ (by Theorem 1.6(vi)).

Case 2: $\mathbf{M} \neq \mathbf{V}$. If $x \in \mathbf{V} \setminus \mathbf{M}$, then $x \notin V_\alpha$ for some inaccessible cardinal α. Thus, $\mathbf{V}$ contains inaccessible cardinals. Let κ be the least inaccessible cardinal in $\mathbf{V}$. Now argue:

- $\mathbf{M} = V_\kappa$.

By Corollary 2.7, $\mathbf{M} = V_\kappa$ is a model of ZFC. To prove that [There exist no inaccessible cardinals] also holds in $\mathbf{M} = V_\kappa$, we proceed by contradiction. Assume that

$$\mathsf{ZFC} \vdash \left(\exists x\, (x \text{ is an inaccessible cardinal})\right)^{V_\kappa}.$$

Fix an $x \in V_\kappa$ such that

$$(x \text{ is an inaccessible cardinal})^{V_\kappa}, \tag{2}$$

and prove that ZFC also proves the following:

- x is an ordinal and $\omega \in x$; this is by absoluteness, applied to the consequence $(x \text{ is an ordinal and } \omega \in x)^{V_\kappa}$ of (2);
- x is a cardinal; otherwise, $\exists y < x\, \exists f\, (f \text{ is a bijection } y \to x)$ where
$$f \in \mathcal{P}(x \times x) \in V_\kappa,$$
so by absoluteness, $\left(\exists y < x\, \exists f\, (f \text{ is a bijection } y \to x)\right)^{V_\kappa}$, contradicting (2);
- x is regular; otherwise, $\exists y < x\, \exists \text{injection } f : y \to x\, (\operatorname{rng}(f) \text{ is unbounded in } x)$ where $f \in \mathcal{P}(x \times x) \in V_\kappa$, so by absoluteness again, we have that $\left(\exists y < x\, \exists \text{injection } f : y \to x\, (\operatorname{rng}(f) \text{ is unbounded in } x)\right)^{V_\kappa}$, contradicting (2);

Thus x is an uncountable regular cardinal. By the choice of κ, $2^y \geq x$ for some cardinal $y < x$, so $\mathsf{ZFC} \vdash \exists y < x\, \exists g\, (g \text{ is an injection } x \to \mathcal{P}(y))$. Here, again, $g \in V_\kappa$. Hence, by absoluteness results, we get that $\mathsf{ZFC} \vdash \left(\exists y < x\, \exists g\, (g \text{ is an injection } x \to \mathcal{P}(y))\right)^{V_\kappa}$, contradicting (2). $\square$

5. The Mostowski Collapse

The Mostowski Collapse is a procedure for obtaining from models (A, R) of ZFC (or certain subsets of ZFC) where R is not real membership, isomorphic models $(M, \in)$ where the relation is membership.

Theorem 5.1. *Let* $\mathbf{R}$ *be a class relation that is well-founded and set-like on a class* $\mathbf{A}$*. There exists a unique class function* $\mathbf{F}\colon \mathbf{A} \to \mathbf{V}$ *such that*

$$\mathbf{F}(a) = \{\mathbf{F}(b) : b \in \mathbf{A},\ (b, a) \in \mathbf{R}\} \quad \textit{for all } a \in \mathbf{A}.$$

Sketch of Proof. For the existence, use the Recursion Theorem for the class function $\mathbf{G}\colon \mathbf{A} \times \mathbf{V} \to \mathbf{V}$ defined for all $a \in \mathbf{A}$ and $x \in \mathbf{V}$ by

$$\mathbf{G}(a, x) = \begin{cases} \operatorname{rng}(x) & \text{if } x \text{ is a function with domain } \operatorname{pred}_{\mathbf{A},\mathbf{R}}(a), \\ \emptyset & \text{otherwise.} \end{cases}$$

For the uniqueness, proceed by contradiction, and consider an $\mathbf{R}$-minimal element of the class where the two class functions differ. $\square$

Definition 5.2. For $\mathbf{A}$ and $\mathbf{R}$ as in Theorem 5.1, the class function $\mathbf{F}$ from the theorem will be referred to as *the Mostowski collapsing function for* $\mathbf{A}$, $\mathbf{R}$, and the range of $\mathbf{F}$ will be called *the Mostowski collapse of* $\mathbf{A}$, $\mathbf{R}$.

Definition 5.3. A class relation $\mathbf{R}$ is said to be *extensional* on a class $\mathbf{A}$ if the following analog of the Extensionality Axiom holds:

$$\mathsf{ZFC} \ \vdash\ \forall x, y \in \mathbf{A}\, \big(\forall z \in \mathbf{A}\, ((z, x) \in \mathbf{R} \leftrightarrow (z, y) \in \mathbf{R})) \to x = y\big).$$

Lemma 5.4. *Let* $\mathbf{R}$ *be a class relation that is well-founded and set-like on a class* $\mathbf{A}$*, and let* $\mathbf{F}$*,* $\mathbf{M}$ *be the Mostowski collapsing function and the Mostowski collapse of* $\mathbf{A}$*,* $\mathbf{R}$*, resp.*

- (i) *For all* $x, y \in \mathbf{A}$*, if* $(x, y) \in \mathbf{R}$ *then* $\mathbf{F}(x) \in \mathbf{F}(y)$*.*
- (ii) $\mathbf{M}$ *is transitive.*
- (iii) *The following conditions are equivalent:*
  - (a) $\mathbf{R}$ *is extensional on* $\mathbf{A}$*;*
  - (b) $\mathbf{F}$ *is one-to-one, and for all* $x, y \in \mathbf{A}$ *we have that* $(x, y) \in \mathbf{R}$ *iff* $\mathbf{F}(x) \in \mathbf{F}(y)$*.*

Sketch of Proof. (i)–(ii) are straightforward. (iii) (b) $\Rightarrow$ (a) follows, because by the assumption in (b), $\mathbf{F}$ is a class isomorphism from $(\mathbf{A}, \mathbf{R})$ onto $(\mathbf{M}, \in)$.

(iii) (a) $\Rightarrow$ (b): The statement follows easily if $\mathbf{F}$ is one-to-one. Assuming $\mathbf{F}$ is not one-to-one, choose x $\mathbf{R}$-minimal so that $\mathbf{F}(x) = \mathbf{F}(y)$ for some $y \neq x$. Then $\{\mathbf{F}(z) : (z, x) \in \mathbf{R}\} = \mathbf{F}(x) = \mathbf{F}(y) = \{\mathbf{F}(u) : (u, y) \in \mathbf{R}\}$ yields a counterexample to (a). $\square$

Theorem 5.5. *Let* $\mathbf{R}$ *be a class relation that is well-founded, set-like, and extensional on a class* $\mathbf{A}$. *Then there exist unique* $\mathbf{F}$ *and* $\mathbf{M}$ *such that* $\mathbf{M}$ *is a transitive class and* $\mathbf{F}$ *is a class isomorphism from* $(\mathbf{A}, \mathbf{R})$ *onto* $(\mathbf{M}, \in)$.

Sketch of Proof. Existence: Let $\mathbf{F}$ and $\mathbf{M}$ be the Mostowski collapsing function and the Mostowski collapse, respectively, and use Lemma 5.4. Uniqueness: Proceed by contradiction, and consider an $\mathbf{R}$-minimal element of the class where the two functions differ. $\square$

6. Reflection Theorems

Our main goal in this section is to prove that if ZFC is consistent, then it has a countable transitive (set) model. This requires studying the following question: Given a (class) model $\mathbf{Z}$ of a finite set of formulas, how can we find a 'small' subset A of $\mathbf{Z}$ such that the given formulas are absolute for A,$\mathbf{Z}$?

Definition and Notation 6.1. Let $\Phi = \{\varphi_0, \ldots, \varphi_{m-1}\}$ be a finite set of formulas. We will say that Φ is *subformula-closed* if for each $i < m$, if ψ is a subformula of φ_i, then $\psi = \varphi_\ell$ for some $\ell < m$. We will use the following notation for any finite, subformula-closed set $\Phi = \{\varphi_0, \ldots, \varphi_{m-1}\}$ of formulas:

- J is the set of all $i < m$ such that φ_i starts with the symbol $\forall$;
- for $i \in J$ we write φ_i in the form $\forall x\, \varphi_j(x, y_1, \ldots, y_t)$ where $x, y_1, \ldots, y_t$ are exactly the free variables of φ_j $(\in \Phi)$ (j, t, $x, y_1, \ldots, y_t$ depend on i).

Lemma 6.2. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be classes such that* $\mathbf{M} \subseteq \mathbf{N}$, *and let* Φ *be a finite, subformula-closed set of formulas. The following conditions on* Φ *are equivalent:*

(a) *Each* $\varphi_i \in \Phi$ *is absolute for* $\mathbf{M}, \mathbf{N}$.

(b) *For each* $i \in J$ *with corresponding formula* $\varphi_i \equiv \forall x\, \varphi_j(x, y_1, \ldots, y_t)$,

$$\mathsf{ZFC} \ \vdash \ \forall y_1, \ldots, y_t \in \mathbf{M} \big(\forall x \in \mathbf{M}\ \varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \to \forall x \in \mathbf{N}\ \varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)\big).$$

Sketch of Proof. (a) $\Rightarrow$ (b): This can be proved by showing that

$$\big\{\forall x, \overline{y} \in \mathbf{M} \big(\varphi_j^{\mathbf{M}}(x, \overline{y}) \leftrightarrow \varphi_j^{\mathbf{N}}(x, \overline{y})\big), \forall \overline{y} \in \mathbf{M} \big(\forall x \in \mathbf{M}\, \varphi_j^{\mathbf{M}}(x, \overline{y}) \leftrightarrow \forall x \in \mathbf{N}\, \varphi_j^{\mathbf{N}}(x, \overline{y})\big)\big\} \\ \vdash \forall \overline{y} \in \mathbf{M} \big(\forall x \in \mathbf{M}\, \varphi_j^{\mathbf{N}}(x, \overline{y}) \to \forall x \in \mathbf{N}\, \varphi_j^{\mathbf{N}}(x, \overline{y})\big).$$

(b) $\Rightarrow$ (a): Since Φ is subformula-closed, one can proceed by induction on the lengths of formulas in Φ. $\square$

The following equivalent formulation of condition (b) in Lemma 6.2 will be useful:

(b)* For each $i \in J$ with corresponding formula $\varphi_i \equiv \forall x\, \varphi_j(x, y_1, \ldots, y_t)$,

$$\mathsf{ZFC} \;\vdash\; \forall y_1, \ldots, y_t \in \mathbf{M}\, \big(\exists x \in \mathbf{N}\, \neg\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \to \exists x \in \mathbf{M}\, \neg\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)\big).$$

Theorem 6.3. *Let $Z\colon \mathbf{On} \to \mathbf{V}$, $\alpha \mapsto Z_\alpha$ be a class function such that*

- *$Z_\gamma \subseteq Z_\alpha$ if $\gamma < \alpha$, and*
- *$Z_\alpha = \bigcup_{\gamma<\alpha} Z_\gamma$ if α is a limit ordinal.*

Let $\mathbf{Z} = \bigcup_{\alpha\in\mathbf{On}} Z_\alpha$. Then for any finite sequence $\varphi_0, \ldots, \varphi_{n-1}$ of formulas,

$$\mathsf{ZFC} \;\vdash\; \forall\alpha \in \mathbf{On}\, \exists\beta \in \mathbf{On} \Big(\beta > \alpha \,\wedge\, \bigwedge_{i<n} [\varphi_i \textit{ is absolute for } Z_\beta, \mathbf{Z}]\Big),$$

where $[\varphi_i$ is absolute for $Z_\beta, \mathbf{Z}]$ is the formula in the definition of "φ_i is absolute for $Z_\beta, \mathbf{Z}$".

Proof. Let Φ be the smallest subformula-closed set of formulas containing $\varphi_0, \ldots, \varphi_{n-1}$. Clearly, Φ is finite. Let $\alpha \in \mathbf{On}$. Our task is to prove the existence of an ordinal $\beta > \alpha$ such that condition (a) of Lemma 6.2 holds with $\mathbf{M} = Z_\beta$ and $\mathbf{N} = \mathbf{Z}$. Actually, we will work with the equivalent condition (b)*.

For each $i \in J$ with corresponding formula $\varphi_i \equiv \forall x\, \varphi_j(x, y_1, \ldots, y_t)$, define class functions

- $\mathbf{G}_i\colon \mathbf{Z} \times \cdots \times \mathbf{Z} \to \mathbf{On}$ for $y_1, \ldots, y_t \in \mathbf{Z}$ by

$$\mathbf{G}_i(y_1, \ldots, y_t) = \begin{cases} \text{least } \eta \in \mathbf{On} \text{ such that } \exists x \in Z_\eta\, \neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t) & \text{if such } \eta \text{ exists,} \\ 0 & \text{otherwise;} \end{cases}$$

- $\mathbf{F}_i\colon \mathbf{On} \to \mathbf{On}$ for $\xi \in \mathbf{On}$ by

$$\mathbf{F}_i(\xi) = \bigcup\{\mathbf{G}_i(y_1, \ldots, y_t) : y_1, \ldots, y_t \in Z_\xi\};$$

- $\gamma\colon \omega \to \mathbf{On}$, $p \mapsto \gamma_p$, using recursion, by $\gamma_0 = \alpha + 1$ and for any $p \in \omega$ by

$$\gamma_{p+1} = \max\Big(\gamma_p + 1,\; \bigcup\{\mathbf{F}_i(\xi) : i \in J,\; \xi \le \gamma_p\} + 1\Big).$$

Let $\beta = \bigcup_{p\in\omega} \gamma_p$. Then:

- $\beta > \alpha$, β is a limit ordinal, and $Z_\beta = \bigcup_{p\in\omega} Z_{\gamma_p}$.

- If $i \in J$, $y_1, \ldots, y_t \in Z_\beta$, and $\exists x \in \mathbf{Z}\, \neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, then
  - there exists $p \in \omega$ such that $y_1, \ldots, y_t \in Z_{\gamma_p}$,
  - $\mathbf{G}_i(y_1, \ldots, y_t) \le \mathbf{F}_i(\gamma_p) < \gamma_{p+1} < \beta$, so
  - $\exists x \in Z_\beta\, \neg\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, proving (b)* with $\mathbf{M} = Z_\beta$ and $\mathbf{N} = \mathbf{Z}$.

$\square$

By applying Theorem 6.3 to $\mathbf{Z} = \mathbf{V}$ (hence, $Z_\alpha = V_\alpha$ for all $\alpha\mathbf{On}$) we get the following.

Corollary 6.4. (Reflection Theorem) *For any finite sequence $\varphi_0, \ldots, \varphi_{n-1}$ of formulas,*

$$\mathsf{ZFC} \vdash \forall \alpha \in \mathbf{On}\, \exists \beta \in \mathbf{On} \Big(\beta > \alpha \wedge \bigwedge_{i<n} [\varphi_i \textit{ is absolute for } V_\beta] \Big).$$

Theorem 6.5. *For any class $\mathbf{Z}$ and for any finite sequence $\varphi_0, \ldots, \varphi_{n-1}$ of formulas,*

$$\mathsf{ZFC} \vdash \forall X \subseteq \mathbf{Z}\, \exists A \subseteq \mathbf{Z} \Big(X \subseteq A \wedge |A| \le \max(\omega, |X|) \wedge \bigwedge_{i<n} [\varphi_i \textit{ is absolute for } A, \mathbf{Z}] \Big).$$

Proof. Let Φ be as before. For each $\gamma \in \mathbf{On}$ let $Z_\gamma = \mathbf{Z} \cap V_\gamma$. Consider a set $X \subseteq \mathbf{Z}$. Since $X \subseteq V_\alpha$ for some α, we get $X \subseteq Z_\alpha$. By Theorem 6.3, there exists $\beta > \alpha$ such that condition (b)* holds with $\mathbf{M} = Z_\beta$ and $\mathbf{N} = \mathbf{Z}$. Our goal is to find $A \subseteq Z_\beta$ with $X \subseteq A$ and $|A| \le \max(\omega, |X|)$ such that (b)* also holds with $\mathbf{M} = A$ and $\mathbf{N} = \mathbf{Z}$.

Choose a well-ordering $\prec$ of Z_β, and for each $i \in J$ with corresponding formula $\varphi_i \equiv \forall x\, \varphi_j(x, y_1, \ldots, y_t)$, define a function $H_i \colon Z_\beta \times \cdots \times Z_\beta \to Z_\beta$ for $y_1, \ldots, y_t \in Z_\beta$ by

$$H_i(y_1, \ldots, y_t) = \begin{cases} \text{the } \prec\text{-least } x \in Z_\beta \text{ such that } \neg\varphi_j^{Z_\beta}(x, y_1, \ldots, y_t) & \text{if such } x \text{ exists,} \\ \text{the } \prec\text{-least element of } Z_\beta & \text{otherwise.} \end{cases}$$

Let A be the least subset of Z_β such that $X \subseteq A$ and A is closed under all functions H_i ($i \in J$). It is easy to see that

- $|A| \le \max(\omega, |X|)$, and

- (b)* holds with $\mathbf{M} = A$ and $\mathbf{N} = \mathbf{Z}$.

$\square$

Theorem 6.6. *For any transitive class* $\mathbf{Z}$ *and for any finite sequence* $\varphi_0, \ldots, \varphi_{n-1}$ *of sentences (!),*

$$(\dagger) \qquad \mathsf{ZFC} \ \vdash \ \forall \text{transitive}\, X \subseteq \mathbf{Z}\, \exists \,\text{transitive}\, M \ \Big(X \subseteq M \wedge |M| \leq \max(\omega, |X|) \wedge \bigwedge_{i<n} (\varphi_i^M \leftrightarrow \varphi_i^{\mathbf{Z}}) \Big).$$

Proof. We may assume that Ext (= Extensionality Axiom) is among $\varphi_0, \ldots, \varphi_{n-1}$. Consider a transitive set $X \subseteq \mathbf{Z}$. By Theorem 6.5, there exists a set $A \subseteq \mathbf{Z}$ which satisfies the condition in (†) (with A in place of M), except that A may not be transitive.

Let F and M be the Mostowski collapsing function and the Mostowski collapse of $A, \in$. By Lemma 5.4, M is transitive and F is an isomorphism from $(A, \in)$ onto $(M, \in)$.

Thus,

- $\mathsf{ZFC} \ \vdash \ \varphi_i^M \leftrightarrow \varphi_i^A$ for all $i < n$, and hence $\mathsf{ZFC} \ \vdash \ \varphi_i^M \leftrightarrow \varphi_i^{\mathbf{Z}}$ for all $i < n$.

Since X is transitive, it follows that

- $F(x) = x$ for all $x \in X$, so $X = F[X] \subseteq F[A] = M$.

This completes the proof of (†). □

Corollary 6.7. *If* S *is a set of sentences containing* ZFC*, then for any* $\varphi_0, \ldots, \varphi_{n-1} \in S$*,*

$$S \ \vdash \ \exists M \ \Big(M \textit{ is transitive} \wedge |M| = \omega \wedge \bigwedge_{i<n} \varphi_i^M \Big).$$

Proof. Apply Theorem 6.6 with $\mathbf{Z} = \mathbf{V}$ and $X = \omega$. □

Corollary 6.7. *If S is a set of sentences containing* ZFC*, then for any $\varphi_0, \dots, \varphi_{n-1} \in S$,*

$$S \ \vdash \ \exists M \ \Big(M \ \textit{is transitive} \wedge |M| = \omega \ \wedge \ \bigwedge_{i<n} \varphi_i^M \Big).$$

Theorem 6.8. *Let S be a consistent set of sentences in the language of set theory such that S contains* ZFC*. In the language expanded by a new constant symbol $\mathcal{M}$, the following set of sentences is consistent:*

$$S \cup \{\mathcal{M} \ \textit{is transitive}\} \cup \{|\mathcal{M}| = \omega\} \cup S^{\mathcal{M}}.$$

Proof. Suppose it is not consistent. As S is consistent, $S \cup \{\mathcal{M} \text{ is transitive}\} \cup \{|\mathcal{M}| = \omega\}$ has a model, and is therefore also consistent. But, when we add the formulas in $S^{\mathbf{M}}$ to this set, the new (larger) set is not consistent. Therefore, since formal proofs have finite lengths, it follows that there exist finitely many sentences $\varphi_0, \dots, \varphi_{n-1} \in S$ such that the set

$$S \cup \{\mathcal{M} \text{ is transitive}\} \cup \{|\mathcal{M}| = \omega\} \cup \{\varphi_i^{\mathcal{M}} : i < n\}$$

is not consistent. Thus,

$$S \ \models \ \neg\Big(\mathcal{M} \text{ is transitive } \ \wedge \ |\mathcal{M}| = \omega \ \wedge \ \bigwedge_{i<n} \varphi_i^{\mathcal{M}}\Big),$$

and therefore

$$S \ \models \ \neg\exists M\Big(M \text{ is transitive } \ \wedge \ |M| = \omega \ \wedge \ \bigwedge_{i<n} \varphi_i^{M}\Big),$$

which contradicts Corollary 6.7. ☐

Corollary 6.9. *If* ZFC *is consistent, then it has a countable transitive model (c.t.m.).*

This allows us to work with countable transitive models of ZFC when we prove next that CH is independent of ZFC.