The Main Theorem Cardinal Arithmetic assuming GCH.

For the MTCA, let's say that κ is *reachable by λ from μ* if $\kappa \leq \mu^\lambda$. We will only need this in the case where $\mu < \kappa$, but $\kappa \leq \mu^\lambda$. Say that κ is *reachable by λ from below* if there exists $\mu < \kappa$ such that κ is reachable by λ from μ.

Theorem 1. *(MTCA without GCH) If $\kappa \geq 2$ and $\lambda \geq \aleph_0$ are cardinals, then κ^λ may be computed inductively as follows:*

(1) *If $\kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$.*
(2) *If κ is reachable by λ from $\mu < \kappa$, then $\kappa^\lambda = \mu^\lambda$.*
(3) *If κ is not reachable by λ from below, then ($\lambda < \kappa$) and*
  (a) *$\kappa^\lambda = \kappa^{cf(\kappa)}$ if $cf(\kappa) \leq \lambda < \kappa$, and*
  (b) *$\kappa^\lambda = \kappa$ if $\lambda < cf(\kappa)$.*

Corollary 2. *(MTCA for infinite cardinals under GCH)*

$$\aleph_\alpha^{\aleph_\beta} = \begin{cases} \aleph_{\beta+1} & \textit{if } \aleph_\alpha \leq \aleph_\beta, \\ \aleph_{\alpha+1} & \textit{if } cf(\aleph_\alpha) \leq \aleph_\beta < \aleph_\alpha, \textit{ and} \\ \aleph_\alpha & \textit{if } \aleph_\beta < cf(\aleph_\alpha). \end{cases}$$

Proof. We let $\kappa = \aleph_\alpha$ and $\lambda = \aleph_\beta$ and refer to the MTCA to establish this corollary, as we now explain.

(Case (1) of MTCA.) If $\kappa = \aleph_\alpha \leq \aleph_\beta = \lambda$, then $\aleph_\alpha^{\aleph_\beta} = \kappa^\lambda = 2^\lambda = \lambda^+ = \aleph_{\beta+1}$ by the MTCA+GCH.

Before continuing, let's determine when κ is reachable from below in the remaining cases where $\lambda < \kappa$. Assume that $\mu < \kappa \leq \mu^\lambda$. If $\nu = \max(\lambda, \mu)$, then we have $\nu < \kappa \leq \mu^\lambda \leq \nu^\nu = \nu^+$, where the last equality follows from GCH. Necessarily $\kappa = \mu^\lambda = \nu^+$. The fact that $\kappa = \mu^\lambda$ implies that $\kappa^\lambda = (\mu^\lambda)^\lambda = \mu^\lambda = \kappa$ The fact that $\kappa = \nu^+$ implies that κ is a successor cardinal. This makes κ regular, so $\lambda, \mu < \kappa = \mathrm{cf}(\kappa)$. In particular, if κ is reachable from below, then we must be in the third case of the statement of this corollary ($\aleph_\beta = \lambda < \mathrm{cf}(\kappa) = \mathrm{cf}(\aleph_\alpha)$) and the formula given in the statement of the corollary is correct ($\aleph_\alpha^{\aleph_\beta} = \kappa^\lambda = \kappa = \aleph_\alpha$).

Let's examine the remaining cases under the assumptions that $\lambda < \kappa$ and κ is not reachable from below by λ. This puts us in Case (3) of the MTCA.

(Case (3)(a) of MTCA.) Assume that κ is not reachable by λ from below and that $\mathrm{cf}(\kappa) \leq \lambda < \kappa$. By the MTCA (3)(a) we have $\kappa \leq \kappa^\lambda = \kappa^{\mathrm{cf}(\kappa)} = \gimel(\kappa)$. We have $\kappa < \gimel(\kappa)$ by Kőnig's Corollary and $\gimel(\kappa) = \kappa^{\mathrm{cf}(\kappa)} \leq \kappa^\kappa \leq \kappa^+$ under GCH. Summarizing, $\kappa < \gimel(\kappa) = \kappa^+$, under GCH. By the second sentence of this paragraph, $\kappa^\lambda = \gimel(\kappa)$, so $\aleph_\alpha^{\aleph_\beta} = \kappa^\lambda = \kappa^+ = \aleph_{\alpha+1}$ in this case. This agrees with the formula give in the statement of the corollary.

(Case (3)(b)of MTCA.) Assume that κ is not reachable by λ from below and that $\lambda < \mathrm{cf}(\kappa)$. By the MTCA (3)(b) we have $\kappa^\lambda = \kappa$. Hence $\aleph_\alpha^{\aleph_\beta} = \kappa = \aleph_\alpha$ in this case. This agrees with the formula give in the statement of the corollary. $\square$