Representations of Boolean algebras.

Recall that a *filter* on a Boolean algebra $\mathbb{B}$ is a nonempty subset of B that is

(1) closed upward, and
(2) closed under $\wedge$.

A filter F in $\mathbb{B}$ is *proper* if $F \subsetneq B$. A filter is an *ultrafilter* is a maximal proper filter. We showed in class that:

Lemma 1. *A filter F of $\mathbb{B}$ is an ultrafilter if and only if for all $b \in B$ either $b \in F$ or $b' \in F$.*

Theorem 2. (Ultrafilter Lemma) *Let $\mathbb{B}$ be a Boolean algebra.*

(1) (Version 1 - the usual version)
If F is a proper filter of $\mathbb{B}$, then F may be extended to an ultrafilter $U \supseteq F$ of $\mathbb{B}$.
(2) (Version 2)
If F is a proper filter of $\mathbb{B}$ and $b \in B - F$, then F may be extended to an ultrafilter $U \supseteq F$ of $\mathbb{B}$ satisfying $b \in B - U$.
(3) (Version 3)
If F is a proper filter of $\mathbb{B}$, I is an ideal, and $F \cap I = \emptyset$, then then F and I may be extended to an ultrafilter $U \supseteq F$ and a prime ideal $P \supseteq I$ of $\mathbb{B}$ such that $U \cap P = \emptyset$. (Necessarily U and P are complementary.)

Proof. I will write the proof in the language of Boolean rings instead of the language of Boolean algebras. I will also replace "(ultra)filters" with the dual concept of "(maximal) ideals". I will also assume that everyone has already seen a proof that if R is a unital ring and I is an ideal of R, then R may be extended to a maximal ideal of R. I will append a proof of this to the end of these notes, in case you haven't seen it before.

In the ring language, our goal is to prove that if R is a Boolean ring, then:

(1) *(Version 1)*
If I is a proper ideal of R, then I may be extended to a maximal ideal $M \supseteq I$ of R.
(2) *(Version 2)*
If I is a proper ideal of R and $b \in R - I$, then I may be extended to a maximal ideal $M \supseteq I$ of R satisfying $b \in R - M$.
(3) *(Version 3)*
If I and J are a proper ideals of R, and $I \cap (1 + J) = \emptyset$, then there is a maximal ideal M such that $I \subseteq M$ and $1 + J \subseteq 1 + M$.

The proofs begin here.

(1) *(Version 1)*
I assume that everyone has seen the proof, using Zorn's Lemma or some other form of the Axiom of Choice, that every proper ideal in a unital ring may be extended to a maximal ideal. (Theorem 4.) $\square$

(2) *(Version 3)*

If $I \cap (1 + J) = \emptyset$, then I claim that $I + J$ is a proper ideal of R. To see this, assume the contrary that $I + J = R \ni 1$. There exist $i \in I$ and $j \in J$ such that $1 = i + j$. Since we are in a Boolean ring, $1 + j = i + j + j = i \in I \cap (1 + J) = \emptyset$, a contradiction.

Now, since $I + J$ is proper, there is a maximal ideal $M \supseteq I + J$ by Version 1 of this theorem. For this M we have $I \subseteq M$ and $1 + J \subseteq 1 + M$.

(3) *(Version 2)*

If I is a proper ideal of the Boolean ring R, then it is a proper ideal of the Boolean algebra. The principal filter $F = [b)$ is a nonempty filter of the Boolean algebra that is disjoint from I. Let $J = 1 + F$ be the ring ideal complementary to F, so that $F = 1 + J$ in the ring R. By Version 3, there is a maximal ideal $M \supseteq I$ such that $F = 1 + J \subseteq 1 + M$. Since $b \in F \subseteq 1 + M$, we have $b \notin M$.

□

If $\mathbb{B}$ is a BA and $b \in B$, let $\mathrm{ult}(b)$ be the set of ultrafilters of $\mathbb{B}$ that contain b.

Theorem 3. *Let $\mathbb{B}$ be a BA and let X be the set of all ultrafilters of $\mathbb{B}$. The function*

$$ult\colon \mathbb{B} \to \mathscr{P}(X) : b \mapsto ult(b)$$

is an embedding.

Proof.

(1) $\mathrm{ult}(0) = \emptyset$.

No ultrafilter contains 0, since every ultrafilter is proper.

(2) $\mathrm{ult}(1) = X$.

Every ultrafilter contains 1, since (ultra)filters are nonempty and closed upward.

(3) $\mathrm{ult}(b') = X - \mathrm{ult}(b)$.

This asserts that every ultrafilter $U \in X$ contains b or b', but not both. By Lemma 1, if $U \in X$, then $b \in U$ or $b' \in U$. U cannot contain both b and b', since this leads to $0 = b \wedge b' \in U$, contradicting the fact that U is proper.

(4) $\mathrm{ult}(c \wedge d) = \mathrm{ult}(c) \cap \mathrm{ult}(d)$.

Since $c \wedge d \leq c, d$, any (ultra)filter containing $c \wedge d$ will contain c and d. Hence $\mathrm{ult}(c \wedge d) \subseteq \mathrm{ult}(c) \cap \mathrm{ult}(d)$. Conversely, if $U \in \mathrm{ult}(c) \cap \mathrm{ult}(d)$, then $c, d \in U$, so $c \wedge d \in U$, so $U \in \mathrm{ult}(c \wedge d)$.

(5) $\mathrm{ult}(c \vee d) = \mathrm{ult}(c) \cup \mathrm{ult}(d)$.

This follows from what is above by De Morgan's Laws. (Any function that preserves meet and negation will preserve join.)

(6) ult is injective.

We must argue that $a \neq b$ implies $\mathrm{ult}(a) \neq \mathrm{ult}(b)$. Choose $a \neq b$ in $\mathbb{B}$. Without loss of generality, assume that $a \not\leq b$. Let $F = [a)$ be the principal filter generated by a. Since $a \not\leq b$, we get $b \notin F$, so by the Ultrafilter Lemma (Version 2) we may extend F to an ultrafilter $U \supseteq F$ ($\ni a$) satisfying $b \notin U$. Since $U \in \mathrm{ult}(a) - \mathrm{ult}(b)$, we get that $\mathrm{ult}(a) \neq \mathrm{ult}(b)$.

□

Appendix.

Theorem 4. *If R is a unital ring and I is a proper ideal of R, then I may be extended to a maximal ideal of R.*

Proof. Let P be the poset of a proper ideals of R that contain I ordered by inclusion. (Note: an ideal J of R is proper if and only if $1 \notin J$.) P is nonempty, since $I \in P$.

Claim 5. *P is inductively ordered.*

We must show that any (well-ordered) chain in P has an upper bound in P. Assume that $C = (I_\alpha)_{\alpha<\kappa}$ is a chain: $\alpha < \beta$ implies $I_\alpha \subseteq I_\beta$. It is enough to show that $J = \bigcup_{\alpha<\kappa} I_\alpha$ belongs to P, since J contains each I_α as a subset.

(1) J is closed under sum: If $r, s \in J = \bigcup_{\alpha<\kappa} I_\alpha$, then $r \in I_\alpha$, $s \in I_\beta$ for some $\alpha, \beta < \kappa$. We have $r, s \in I_{\max(\alpha,\beta)} \subseteq J$, so $r + s \in I_{\max(\alpha,\beta)} \subseteq J$.
(2) J is closed under multiplication by elements of R: If $r \in R$ and $s \in J = \bigcup_{\alpha<\kappa} I_\alpha$, then $s \in I_\beta$ for some $\beta < \kappa$. We have $rs \in I_\beta \subseteq J$.
(3) J is proper: $1 \notin I_\alpha$ for any $\alpha < \kappa$, so $1 \notin \bigcup_{\alpha<\kappa} I_\alpha = J$. □

Given the claim, we may apply Zorn's Lemma to obtain a maximal element $M \in P$, which is a maximal proper ideal of R containing I. □