Boolean algebras.

Boolean algebras are the algebraic structures we introduce to study the arithmetic of intersection, union and complementation of sets. The concrete structure we have in mind when we introduce BAs is

$$\langle \mathscr{P}(X); \cap, \cup, ', \emptyset, X \rangle$$

where $A \cap B$ represents 'A intersect B', $A \cup B$ represents 'A union B', A' represents the complement of A relative to X (= X - A), \emptyset is a distinguished subset of X (the empty subset) and X is another distinguished subset of X (the whole subset). We formalize this in an algebraic language with nonlogical symbols \wedge and \vee (binary operation symbols), ' (a unary operation symbol), and 0 and 1 (0-ary operation symbols, or "constants").

Definition. A Boolean algebra is an algebraic structure $\langle B; \wedge, \vee, ', 0, 1 \rangle$ satisfying the laws

- (1) Laws of intersection (= laws of \land -semilattices)
 - (a) (Idempotent Law for \wedge) $x \wedge x = x$.
 - (b) (Commutative Law for \wedge) $x \wedge y = y \wedge x$.
 - (c) (Associative Law for \wedge) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.
- (2) Laws of union (= laws of \lor -semilattices)
 - (a) (Idempotent Law for \lor) $x \lor x = x$.
 - (b) (Commutative Law for \lor) $x \lor y = y \lor x$.
 - (c) (Associative Law for \lor) $x \lor (y \lor z) = (x \lor y) \lor z$.
- (3) Interaction between \wedge and \vee .
 - (a) (\wedge -Absorption Law) $x \wedge (x \lor y) = x$.
 - (b) (\lor -Absorption Law) $x \lor (x \land y) = x$.
 - (c) (Distributivity of \land over \lor) $x \land (y \lor z) = (x \land y) \lor (x \land z)$.
 - (d) (Distributivity of \lor over \land) $x \lor (y \land z) = (x \lor y) \land (x \lor z)$.
- (4) Laws of complement.
 - (a) (x')' = x.
- (5) Interaction between complement and \land, \lor .
 - (a) (De Morgan's Laws)
 - (i) $(x \wedge y)' = x' \vee y'$.
 - (ii) $(x \lor y)' = x' \land y'$.
- (6) Laws involving 0, 1.
 - (a) $0 \wedge x = 0$.
 - (b) $0 \lor x = x$
 - (c) $1 \lor x = 1$.
 - (d) $1 \wedge x = x$.
 - (e) 0' = 1.
 - (f) 1' = 0.
 - (g) $x \wedge x' = 0$.
 - (h) $x \lor x' = 1$.

Here a "law" is an atomic formula in an algebraic language.

Examples of BAs.

- (1) The intended model of these axioms is the power set algebra $\langle \mathscr{P}(X); \cap, \cup, ', \emptyset, X \rangle$. We write this as $\mathscr{P}(X)$.
- (2) Let X be an infinite set. The finite-cofinite Boolean algebra on X is the subalgebra $\operatorname{FinCo}(X)$ of $\mathscr{P}(X)$ consisting of the finite and cofinite subsets of X. (Check that this is a subalgebra!)
- (3) Let X be an infinite set. $\mathscr{P}(X)$ /fin is the quotient of $\mathscr{P}(X)$ obtained by identifying two sets $U, V \in \mathscr{P}(X)$ if U and V differ in only fnitely many elements. (I.e., if the sets U V and V U are both finite.)
- (4) Let S be a set of propositional letters and let P be the set of all propositions in these letters. Define two propositions to be equivalent if they are logically equivalent (equivalently, if they have the same truth table). Write $[\alpha]$ for the equivalence class of a proposition $\alpha \in P$ and let [P] be the set of all equivalence classes. Define $[\alpha] \wedge [\beta] = [\alpha \wedge \beta], \ [\alpha] \vee [\beta] = [\alpha \vee \beta], \ [\alpha]' = [\neg \alpha], \ 1 = [\top], \ 0 = [\bot]$. The resulting BA is $\langle [P]; \wedge, \vee, ', 0, 1 \rangle$.

Remarks.

- (1) Boolean algebras are ordered. (\wedge -order, \vee -order.)
- (2) Principal of Duality. (BAs are self-dual.)
- (3) Boolean algebras versus Boolean rings.
- (4) Complete atomic Boolean algebras are power set algebras.

BA glossary.

- (1) Atoms and coatoms.
- (2) Ideals and filters. (Principal versus nonprincipal.)
- (3) Prime ideals = maximal ideals. Ultrafilters = maximal filters.
- (4) Characters of BAs.
- (5) Ultrafilter lemma.
- (6) Stone duality.
- (7) Complete Boolean algebras.

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