Example of a proof by induction.

Theorem 1. *A subset of a finite set is finite.*

We explained in class why it suffices to prove the following special case.

Theorem 2. *If $k \in \mathbb{N}$, then any subset of k is finite.*

Proof.
Let s_k be the statement "Any subset of k is finite." We will prove this by induction on k.

(Base Case: $k = 0$) If $k = 0 = \emptyset$ and $Y \subseteq k$, then $Y = \emptyset$. In this case, the empty function $f = \emptyset$ is a bijection from $r = 0 \in \mathbb{N}$ to Y.

(Inductive Step: Assume the statement is true for k, prove true for $S(k)$)
Choose any subset $Y \subseteq S(k)$. We must find some $r \in \mathbb{N}$ and a bijection $g : r \to Y$.

Case 1. $Y \subseteq k$.
By the inductive hypothesis, there is an $t \in \mathbb{N}$ and a bijection $f : t \to Y$, so we can take $r = t$ and $g = f$.

Case 2. $Y \not\subseteq k$.
Since $Y \subseteq S(k) = k \cup \{k\}$, if $Y \not\subseteq k$, then $k \in Y$ and $Y - \{k\} \subseteq k$.
Use Case 1 to find $t \in \mathbb{N}$ and a bijection $f : t \to Y - \{k\}$. Now let $r = t \cup \{t\} = S(t) \in \mathbb{N}$ and define a bijection $g : r \to Y$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in t \\ k & \text{if } x = t. \end{cases}$$