Stirling numbers and Bell numbers

Stirling numbers count partitions

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$,

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1,$

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells.

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ {n \atop k} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells. (I use $0/1/2$ as shorthand for this partition.)

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells. (I use $0/1/2$ as shorthand for this partition.)
- $S(3,2) = 3$,

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells. (I use $0/1/2$ as shorthand for this partition.)
- $S(3,2) = 3$,

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells. (I use $0/1/2$ as shorthand for this partition.)
- $S(3,2) = 3$, since we have $01/2$,

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells. (I use $0/1/2$ as shorthand for this partition.)
- $S(3,2) = 3$, since we have $01/2,\ 02/1,$

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells. (I use $0/1/2$ as shorthand for this partition.)
- $S(3,2) = 3$, since we have $01/2,\ 02/1,\ 12/0$.
- $S(3,1) = 1$,

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells. (I use $0/1/2$ as shorthand for this partition.)
- $S(3,2) = 3$, since we have $01/2,\ 02/1,\ 12/0$.
- $S(3,1) = 1$,

Stirling numbers count partitions

Definition. The number of partitions of an n-element set into k cells is denoted

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} \qquad \text{or} \qquad S(n,k),$$

and is called a *Stirling number of the second kind*.

Examples.

- $S(3,3) = 1$, since $\{\{0\},\{1\},\{2\}\}$ is the only partition of $\{0,1,2\}$ into 3 cells. (I use $0/1/2$ as shorthand for this partition.)
- $S(3,2) = 3$, since we have $01/2,\ 02/1,\ 12/0$.
- $S(3,1) = 1$, since we have only 012.

$S(n, k)$ is "dual" to $C(n, k)$

$S(n, k)$ is “dual” to $C(n, k)$

The number of images of injective functions $k \to n$

$S(n, k)$ is "dual" to $C(n, k)$

The number of images of injective functions $k \to n$ equals the number of k-element subsets of n,

$S(n,k)$ is "dual" to $C(n,k)$

The number of images of injective functions $k \to n$ equals the number of k-element subsets of n, which is counted by the function $C(n,k)$.

$S(n,k)$ is "dual" to $C(n,k)$

The number of images of injective functions $k \to n$ equals the number of k-element subsets of n, which is counted by the function $C(n,k)$.

The number of coimages of surjective functions $n \to k$

$S(n,k)$ is "dual" to $C(n,k)$

The number of images of injective functions $k \to n$ equals the number of k-element subsets of n, which is counted by the function $C(n,k)$.

The number of coimages of surjective functions $n \to k$ equals the number of partitions of n into k cells,

$S(n,k)$ is "dual" to $C(n,k)$

The number of images of injective functions $k \to n$ equals the number of k-element subsets of n, which is counted by the function $C(n,k)$.

The number of coimages of surjective functions $n \to k$ equals the number of partitions of n into k cells, which is counted by the function $S(n,k)$.

$S(n,k)$ is "dual" to $C(n,k)$

The number of images of injective functions $k \to n$ equals the number of k-element subsets of n, which is counted by the function $C(n,k)$.

The number of coimages of surjective functions $n \to k$ equals the number of partitions of n into k cells, which is counted by the function $S(n,k)$.

There are many parallels between $C(n,k)$ and $S(n,k)$.

Recursion

Recursion

Theorem.

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

Theorem.

(1) $S(n, k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n, n) = 1$ and $S(n, 0) = 0$ if $n > 0$.

(3) $S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

$S(n, n) = 1$ since $0/1/2/\cdots/n-1$ is the only partition of n into n cells.

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

$S(n,n) = 1$ since $0/1/2/\cdots/n-1$ is the only partition of n into n cells.

If $n > 0$, then $S(n,0) = 0$, since there can be no partition of n into 0 cells.

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

$S(n,n) = 1$ since $0/1/2/\cdots/n-1$ is the only partition of n into n cells.

If $n > 0$, then $S(n,0) = 0$, since there can be no partition of n into 0 cells. (The union of the cells must be n.)

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

$S(n,n) = 1$ since $0/1/2/\cdots/n-1$ is the only partition of n into n cells.

If $n > 0$, then $S(n,0) = 0$, since there can be no partition of n into 0 cells. (The union of the cells must be n.)

For Item (3), count the number of partitions of $\{x_1, x_2, \ldots, x_n\}$ into k cells by considering which cell gets x_n.

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

$S(n,n) = 1$ since $0/1/2/\cdots/n-1$ is the only partition of n into n cells.

If $n > 0$, then $S(n,0) = 0$, since there can be no partition of n into 0 cells. (The union of the cells must be n.)

For Item (3), count the number of partitions of $\{x_1, x_2, \ldots, x_n\}$ into k cells by considering which cell gets x_n. There are $S(n-1,k-1)$ partitions in which x_n is isolated.

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

$S(n,n) = 1$ since $0/1/2/\cdots/n-1$ is the only partition of n into n cells.

If $n > 0$, then $S(n,0) = 0$, since there can be no partition of n into 0 cells. (The union of the cells must be n.)

For Item (3), count the number of partitions of $\{x_1, x_2, \ldots, x_n\}$ into k cells by considering which cell gets x_n. There are $S(n-1,k-1)$ partitions in which x_n is isolated. (That is, of the form $\cdots/\cdots/\cdots/x_n$.)

Recursion

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

$S(n,n) = 1$ since $0/1/2/\cdots/n-1$ is the only partition of n into n cells.

If $n > 0$, then $S(n,0) = 0$, since there can be no partition of n into 0 cells. (The union of the cells must be n.)

For Item (3), count the number of partitions of $\{x_1, x_2, \ldots, x_n\}$ into k cells by considering which cell gets x_n. There are $S(n-1,k-1)$ partitions in which x_n is isolated. (That is, of the form $\cdots/\cdots/\cdots/x_n$.) There are $C(k,1) \cdot S(n-1,k) = k \cdot S(n-1,k)$ partitions in which x_n is not isolated.

Theorem.

(1) $S(n,k) = 0$ if $k < 0$ or $k > n$.

(2) $S(n,n) = 1$ and $S(n,0) = 0$ if $n > 0$.

(3) $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. Item (1) states that there are no partitions of n into k cells if k is negative or bigger than n.

$S(n,n) = 1$ since $0/1/2/\cdots/n-1$ is the only partition of n into n cells.

If $n > 0$, then $S(n,0) = 0$, since there can be no partition of n into 0 cells. (The union of the cells must be n.)

For Item (3), count the number of partitions of $\{x_1, x_2, \ldots, x_n\}$ into k cells by considering which cell gets x_n. There are $S(n-1,k-1)$ partitions in which x_n is isolated. (That is, of the form $\cdots/\cdots/\cdots/x_n$.) There are $C(k,1) \cdot S(n-1,k) = k \cdot S(n-1,k)$ partitions in which x_n is not isolated. □

Binomial-type theorems

Binomial Theorem.

Binomial-type theorems

Binomial Theorem.
$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Binomial-type theorems

Binomial Theorem.
$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$

Stirling Binomial-type Theorem.

Binomial-type theorems

Binomial Theorem.
$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$

Stirling Binomial-type Theorem.
$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}},$

Binomial-type theorems

Binomial Theorem.
$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$.

Stirling Binomial-type Theorem.
$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$,
where $x^{\underline{k}} = (x)_k = x(x-1)\cdots(x-(k-1))$.

Table of Stirling numbers of the second kind

Table of Stirling numbers of the second kind

$n \backslash k$	0	1	2	3	4	5	6	7	8	$\cdots$
0	1	0	0	0	0	0	0	0	0	$\cdots$
1	0	1	0	0	0	0	0	0	0	$\cdots$
2	0	1	1	0	0	0	0	0	0	$\cdots$
3	0	1	3	1	0	0	0	0	0	$\cdots$
4	0	1	7	6	1	0	0	0	0	$\cdots$
5	0	1	15	25	10	1	0	0	0	$\cdots$
6	0	1	31	90	65	15	1	0	0	$\cdots$
7	0	1	63	301	350	140	21	1	0	$\cdots$
8	0	1	127	966	1701	1050	266	28	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table of Stirling numbers of the second kind

$n\backslash k$	0	1	2	3	4	5	6	7	8	$\cdots$
0	1	0	0	0	0	0	0	0	0	$\cdots$
1	0	1	0	0	0	0	0	0	0	$\cdots$
2	0	1	1	0	0	0	0	0	0	$\cdots$
3	0	1	3	1	0	0	0	0	0	$\cdots$
4	0	1	7	6	1	0	0	0	0	$\cdots$
5	0	1	15	25	10	1	0	0	0	$\cdots$
6	0	1	31	90	65	15	1	0	0	$\cdots$
7	0	1	63	301	350	140	21	1	0	$\cdots$
8	0	1	127	966	1701	1050	266	28	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Each row is a unimodal sequence with maximum occurring for one or two consecutive values around $k \approx \frac{n}{\ln(n)}$.

Table of Stirling numbers of the second kind

$n\backslash k$	0	1	2	3	4	5	6	7	8	$\cdots$
0	1	0	0	0	0	0	0	0	0	$\cdots$
1	0	1	0	0	0	0	0	0	0	$\cdots$
2	0	1	1	0	0	0	0	0	0	$\cdots$
3	0	1	3	1	0	0	0	0	0	$\cdots$
4	0	1	7	6	1	0	0	0	0	$\cdots$
5	0	1	15	25	10	1	0	0	0	$\cdots$
6	0	1	31	90	65	15	1	0	0	$\cdots$
7	0	1	63	301	350	140	21	1	0	$\cdots$
8	0	1	127	966	1701	1050	266	28	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Each row is a unimodal sequence with maximum occurring for one or two consecutive values around $k \approx \frac{n}{\ln(n)}$.

The n row sum is denoted B_n and is called the nth **Bell number**.

The Bell numbers

Definition.

The Bell numbers

Definition. B_n is the number of partitions of n.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.
$B_3 = 5$, since the partitions of $3 = \{0, 1, 2\}$ are

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.
$B_3 = 5$, since the partitions of $3 = \{0, 1, 2\}$ are

$$0/1/2,\quad 01/2,\quad 02/1,\quad 12/0,\quad 012.$$

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.
$B_3 = 5$, since the partitions of $3 = \{0, 1, 2\}$ are

$$0/1/2,\quad 01/2,\quad 02/1,\quad 12/0,\quad 012.$$

We have seen that $B_n = \sum_{k=0}^{n} S(n, k)$.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.
$B_3 = 5$, since the partitions of $3 = \{0, 1, 2\}$ are

$$0/1/2, \quad 01/2, \quad 02/1, \quad 12/0, \quad 012.$$

We have seen that $B_n = \sum_{k=0}^{n} S(n, k)$.

Another interesting relation is $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot B_k$.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.
$B_3 = 5$, since the partitions of $3 = \{0, 1, 2\}$ are

$$0/1/2,\quad 01/2,\quad 02/1,\quad 12/0,\quad 012.$$

We have seen that $B_n = \sum_{k=0}^{n} S(n, k)$.

Another interesting relation is $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot B_k$.

Proof. Let $X = \{x_1, x_2, \ldots, x_{n+1}\}$.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.
$B_3 = 5$, since the partitions of $3 = \{0, 1, 2\}$ are

$$0/1/2, \quad 01/2, \quad 02/1, \quad 12/0, \quad 012.$$

We have seen that $B_n = \sum_{k=0}^{n} S(n, k)$.

Another interesting relation is $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot B_k$.

Proof. Let $X = \{x_1, x_2, \ldots, x_{n+1}\}$. A partition of X is determined by the choice of the cell $[x_{n+1}]$ (= a subset of X containing x_{n+1}) and a partition of $X - [x_{n+1}]$.

The Bell numbers

Definition. B_n is the number of partitions of n.

Example. $B_0 = 1$, since the only partition of $0 = \emptyset$ is $\emptyset$.
$B_1 = 1$, since the only partition of $1 = \{0\}$ is $\{\{0\}\}$.
$B_2 = 2$, since the only partitions of $2 = \{0, 1\}$ are $0/1,\ 01$.
$B_3 = 5$, since the partitions of $3 = \{0, 1, 2\}$ are

$$0/1/2,\quad 01/2,\quad 02/1,\quad 12/0,\quad 012.$$

We have seen that $B_n = \sum_{k=0}^{n} S(n, k)$.

Another interesting relation is $B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \cdot B_k$.

Proof. Let $X = \{x_1, x_2, \ldots, x_{n+1}\}$. A partition of X is determined by the choice of the cell $[x_{n+1}]$ (= a subset of X containing x_{n+1}) and a partition of $X - [x_{n+1}]$. $\square$

Growth rates

Growth rates

Which of the following functions grows faster?

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer.

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order.

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \leq B_n \leq n! \leq n^n \leq 2^{n^2}.$$

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \leq B_n \leq n! \leq n^n \leq 2^{n^2}.$$

- $2^{n-1} \leq B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \le B_n \le n! \le n^n \le 2^{n^2}.$$

- $2^{n-1} \le B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \leq B_n \leq n! \leq n^n \leq 2^{n^2}.$$

- $2^{n-1} \leq B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.
- $B_n \leq n!$, since we can code a partition as a permutation.

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \le B_n \le n! \le n^n \le 2^{n^2}.$$

- $2^{n-1} \le B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.
- $B_n \le n!$, since we can code a partition as a permutation.

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \leq B_n \leq n! \leq n^n \leq 2^{n^2}.$$

- $2^{n-1} \leq B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.
- $B_n \leq n!$, since we can code a partition as a permutation. Linearly order cells by least element and linearly order elements of a cell by reverse natural order.

Growth rates

Which of the following functions grows faster?

$$2^{n-1},\ \ B_n,\ \ n!,\ \ n^n,\ \ 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \leq B_n \leq n! \leq n^n \leq 2^{n^2}.$$

- $2^{n-1} \leq B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.
- $B_n \leq n!$, since we can code a partition as a permutation. Linearly order cells by least element and linearly order elements of a cell by reverse natural order.

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \leq B_n \leq n! \leq n^n \leq 2^{n^2}.$$

- $2^{n-1} \leq B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.
- $B_n \leq n!$, since we can code a partition as a permutation. Linearly order cells by least element and linearly order elements of a cell by reverse natural order.
- $n! \leq n^n$, since the latter counts the number of functions $f : n \to n$, while the former only counts the bijections.

Growth rates

Which of the following functions grows faster?

$$2^{n-1}, \quad B_n, \quad n!, \quad n^n, \quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \leq B_n \leq n! \leq n^n \leq 2^{n^2}.$$

- $2^{n-1} \leq B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.
- $B_n \leq n!$, since we can code a partition as a permutation. Linearly order cells by least element and linearly order elements of a cell by reverse natural order.
- $n! \leq n^n$, since the latter counts the number of functions $f : n \to n$, while the former only counts the bijections.

Growth rates

Which of the following functions grows faster?

$$2^{n-1},\quad B_n,\quad n!,\quad n^n,\quad 2^{n^2}$$

Answer. The functions are already in the appropriate order. That is,

$$2^{n-1} \leq B_n \leq n! \leq n^n \leq 2^{n^2}.$$

- $2^{n-1} \leq B_n$, since the latter counts the number of all partitions of n, while the former counts only the number of partitions of n into at most 2 cells.
- $B_n \leq n!$, since we can code a partition as a permutation. Linearly order cells by least element and linearly order elements of a cell by reverse natural order.
- $n! \leq n^n$, since the latter counts the number of functions $f : n \to n$, while the former only counts the bijections.
- $n^n \leq 2^{n^2}$, since the latter counts the number of binary relations from n to n, while the former only counts the binary relations that are functions.