Vector spaces.

Let $\mathbb{F}$ be a field.

Definition 1. An **$\mathbb{F}$-vector space** (or $\mathbb{F}$-space, or vector space over $\mathbb{F}$) is an algebraic structure $\mathbb{V} = \langle V; +, -, 0, \{r(x) \mid r \in \mathbb{F}\}\rangle$ which satisfies the following

(1) Additive laws[1]:
 (a) (Associative law) $\forall x \forall y \forall z((x + (y + z)) = ((x + y) + z))$.
 (b) (Commutative law) $\forall x \forall y(x + y = y + x)$.
 (c) (Unit law) $\forall x(x + 0 = x)$
 (d) (Inverse law) $\forall x(x + (-x) = 0)$

(2) Scaling laws:
 (a) (Scaling by r is an additive endomorphism)
  (i) $\forall x \forall y(r(x + y) = r(x) + r(y))$.
  (ii) $\forall x(r(-x) = -r(x))$.
  (iii) $r(0) = 0$.
 (b) (Scaling is an $\mathbb{F}$-action)
  (i) $\forall x((r + s)(x) = r(x) + s(x))$.
  (ii) $\forall x((-r)(x) = -(r(x))$.
  (iii) $\forall x(0(x) = 0)$.
  (iv) $\forall x((r \cdot s)(x) = r(s(x))$.
  (v) $\forall x(1(x) = x)$.

Examples.

(1) (Most important example!) $\mathbb{V} = \mathbb{R}^m$ is an $\mathbb{R}$-space.
(2) $\mathbb{V} = \mathbb{R}[x]$ = polynomials in the variable x with coefficients in $\mathbb{R}$.
(3) $\mathbb{V} = M_{m\times n}(\mathbb{R}) = m \times n$-matrices with entries in $\mathbb{R}$.
(4) Same type of examples as above with the field $\mathbb{R}$ replaced by some other field $\mathbb{F}$. (That is, $\mathbb{F}^m$, $\mathbb{F}[x]$, $M_{m\times n}(\mathbb{F})$.)

The following concepts are defined for arbitrary vector spaces exactly as they were defined for the $\mathbb{R}$-space $\mathbb{V} = \mathbb{R}^m$:

(1) linear combination
(2) span, or linear closure
(3) linearly independent
(4) basis
(5) dimension
(6) subspace

[1] A **law** or **identity** is a universally quantified equation.