Image and Kernel, Rank and Nullity

Image and Kernel, picture

Definitions.
Let $T\colon \mathbb{V} \to \mathbb{W}$ be linear and let A be a matrix for T (so $T(\mathbf{x}) = A \cdot \mathbf{x}$).

$$\begin{aligned} \operatorname{im}(T) &= \{\mathbf{b} \in \mathbb{W} \mid (\exists \mathbf{x})(T(\mathbf{x}) = \mathbf{b})\} \\ &= \{\mathbf{b} \in \mathbb{W} \mid (\exists \mathbf{x})(A\mathbf{x} = \mathbf{b})\} \\ &= \textbf{column space} \text{ of } A \\ &= \text{a subspace of } \mathbb{W} \end{aligned}$$

$\dim(\operatorname{im}(T))$ = **rank** of T = **rank**(A).

$$\begin{aligned} \ker(T) &= \{\mathbf{x} \in \mathbb{V} \mid T(\mathbf{x}) = \mathbf{0}\} \\ &= \{\mathbf{x} \in \mathbb{V} \mid A\mathbf{x} = \mathbf{0}\} \\ &= \textbf{null space} \text{ of } A \\ &= \text{a subspace of } \mathbb{V} \end{aligned}$$

$\dim(\ker(T))$ = **nullity** of T = **nullity**(A).

Example

Define $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A \cdot \mathbf{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 3 & 7 & 11 \\ 2 & 4 & 6 & 8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.

Question. What can we say about the image and kernel?

Algorithm to compute a basis for $\mathrm{im}(T)$

Let $T(\mathbf{x}) = A \cdot \mathbf{x}$.

Algorithm 1.
If the RRE form of A has pivots in columns $i_1, \ldots, i_r$, then the corresponding columns of A form a basis for $\mathrm{col}(A) = \mathrm{im}(T)$. Hence $\mathrm{rank}(A) = r$.

Algorithm to compute a basis for $\ker(T)$

Algorithm 2.
If the solution set to $A\mathbf{x} = \mathbf{0}$ has the form $x_{j_1}\mathbf{v}_{j_1} + \cdots + x_{j_s}\mathbf{v}_{j_s}$, where $x_{j_1}, \ldots, x_{j_s}$ are the free variables of the homogeneous system, then $\{\mathbf{v}_{j_1}, \ldots, \mathbf{v}_{j_s}\}$ is a basis for $\text{null}(A) = \ker(T)$. Hence $\text{nullity}(A) = s$.

Algorithm to extend a partial basis to a basis

Suppose that $\mathcal{B}$ is a basis for $\mathbb{V}$ and $\mathbb{D}$ is a partial basis (= linearly independent set). How do we find a basis for $\mathbb{V}$ that contains $\mathcal{D}$?

Algorithm 3.
Apply the column space algorithm to $[\mathcal{D} \mid \mathcal{B}]$.

Other algorithms

Suppose that $\mathbb{V}$ has basis $\mathcal{B}$ and subspaces $S, T \leq \mathbb{V}$, which have bases $\mathcal{C}$ and $\mathcal{D}$ respectively.

How do you find bases for $S \cap T$, $S + T$, and S' = some complement to S?

What can you say about the dimensions of these spaces?