Image and Kernel, Rank and Nullity

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Proof.

$$\mathrm{rank}(A) + \mathrm{nullity}(A) \quad = \text{number of pivot cols.} + \text{number of non-pivot cols.}$$

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$$\begin{aligned} \mathrm{rank}(A) + \mathrm{nullity}(A) &= \text{number of pivot cols.} + \text{number of non-pivot cols.} \\ &= \text{number of columns of } A \end{aligned}$$

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Example

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Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A \cdot \mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

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$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

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First column of RRE form is the only pivot column of A, so $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is an ordered basis for $\mathrm{im}(T)$

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First column of RRE form is the only pivot column of A, so $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is an ordered basis for $\mathrm{im}(T)$ (= $\mathrm{col}(A)$), and $\mathrm{rank}(T) = 1$.

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Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A \cdot \mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

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From the computation in Algorithm 1, x_2 is the only free variable,

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First column of RRE form is the only pivot column of A, so $\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ is an ordered basis for $\mathrm{im}(T)$ (= $\mathrm{col}(A)$), and $\mathrm{rank}(T) = 1$.

Algorithm 2. ($\ker(T)$ or $\mathrm{null}(A)$)
From the computation in Algorithm 1, x_2 is the only free variable, and the solution to the homogeneous system is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Example

Define $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A \cdot \mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

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