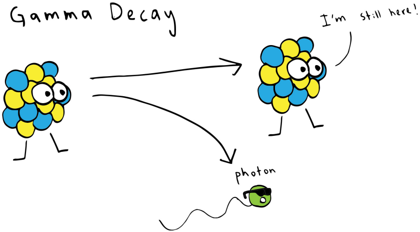
Truth versus Provability



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So, a "proof system" typically specifies its axioms and also the accepted rules of deduction.

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Another way to think about this is: at the first-order level, every statement has a proof or a counterexample.

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$$(\forall n)(\forall f)(\neg(f\colon n \to X \text{ is a bijection})).$$

Then $\Sigma \models Q$, but $\Sigma \not\vdash Q$ for any proof system requiring finite-length proofs.

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Assume that $0 < x < 1$. Since $0 < x$, multiplication by x preserves inequalities. Multiply $x < 1$ by x to obtain $x^2 < x$. $\square$

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Assume that $0 < x < 1$ and $x^2 \not< x$.

Examples!

Let's try these three forms of proof on a theorem concerning $\mathbb{R}$.

Theorem. If $0 < x < 1$, then $x^2 < x$.

Proof structure #1.
Assume that $0 < x < 1$. Since $0 < x$, multiplication by x preserves inequalities. Multiply $x < 1$ by x to obtain $x^2 < x$. $\square$

Proof structure #2.
Assume that $x^2 \nless x$. Then $x \leq x^2$. Hence $0 \leq x^2 - x = x(x-1)$. Hence $0 \leq x, x-1$ or $x, x-1 \leq 0$. The first leads to $0 \leq x - 1$, or $1 \leq x$, while the second leads to $x \leq 0$. Either way, $0 < x < 1$ fails.

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- $\exists$ chooses $x = r$. Wins, because the side game guarantees that $\varphi(r)$ holds.

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