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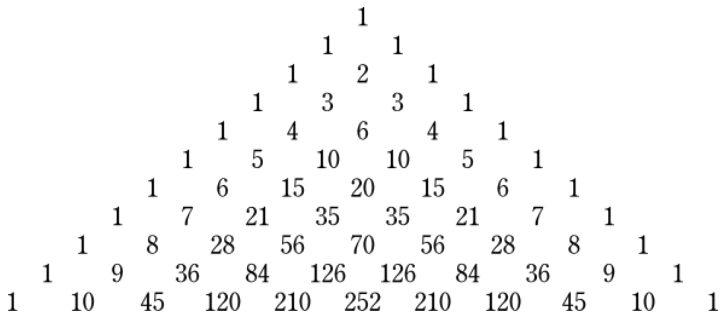
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For an alternative proof, use the formula.

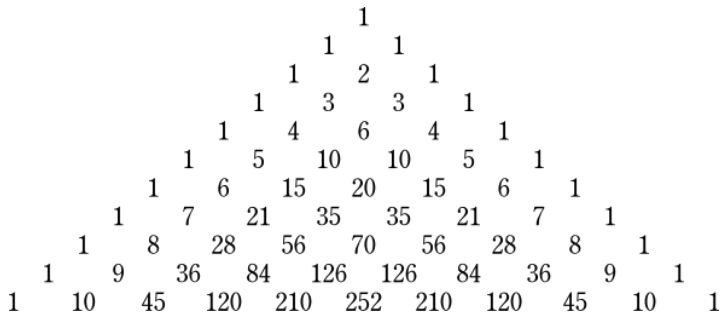
Pascal's Triangle

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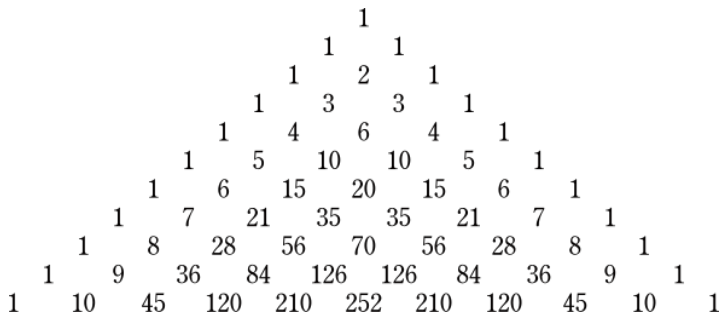


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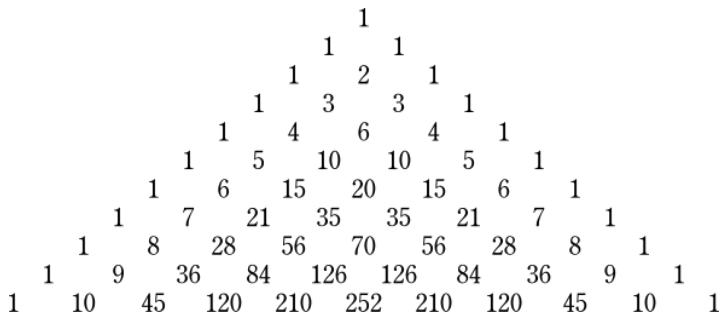
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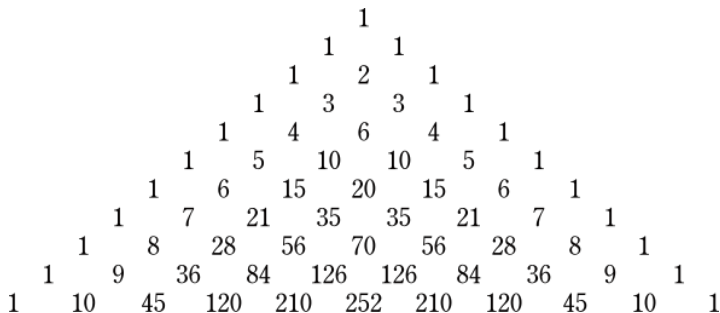
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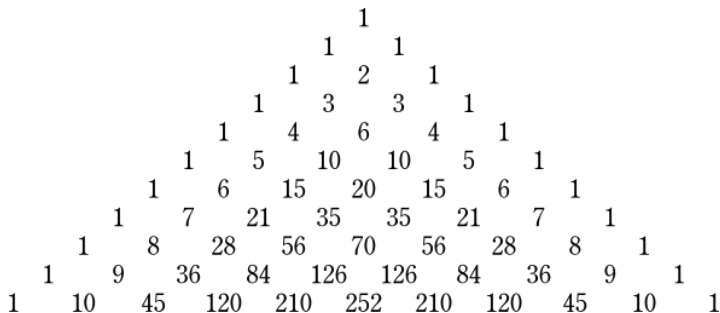
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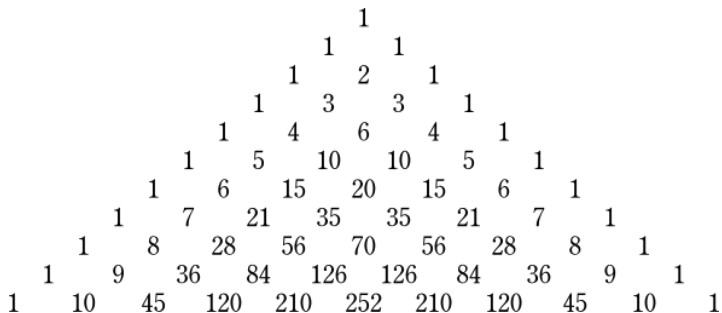


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Unimodal, since $1 \leq \binom{n}{k+1} / \binom{n}{k} = \frac{k!(n-k)!}{(k+1)!(n-k-1)!} = \frac{n-k}{k+1} \Leftrightarrow k \leq \frac{n-1}{2}$.

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Assume true for n , prove it for $n + 1$. Multiply (*) by $x + y$, use IH:

$$\begin{aligned} (x + y)^{n+1} &= \binom{n}{0}x^{n+1}y^0 &+& \binom{n}{1}x^ny^1 &+& \cdots &+& \binom{n}{n}x^1y^n \\ &&+& \binom{n}{0}x^ny^1 &+& \cdots &+& \binom{n}{n-1}x^1y^n &+& \binom{n}{n}x^0y^{n+1} \end{aligned}$$

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$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \dots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$$

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“Stars and Bars” Proof.

[Whiteboard!]

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