LINEAR ALGEBRA
MIDTERM

Name:____________________________

You have 80 minutes for this exam. If you have a question, raise your hand and remain seated. In order to receive full credit your answer must be **complete**, **legible** and **correct**.

(1) True or False. Give a brief (1-sentence) justification or counterexample.

(a) Every homogeneous system is consistent.

True. Any homogeneous system $A\mathbf{x} = \mathbf{0}$ has $\mathbf{x} = \mathbf{0}$ as a solution.

(b) If the columns of a matrix A are linearly independent, then the rows of A are linearly independent.

False. A counterexample is $A = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

But Statement (b) is Almost True! If A is a square matrix whose columns are independent, then its rows will be independent.

(c) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent if the vector $\mathbf{u}$ is not a linear combination of the others in the set.

False. A counterexample in $\mathbb{R}^2$ is $\mathbf{u} = \mathbf{e}_1$ and $\mathbf{v} = \mathbf{w} = \mathbf{e}_2$.

But Statement (c) is Almost True! If each one of $\mathbf{u}, \mathbf{v}, \mathbf{w}$ fails to be a linear combination of the others, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ will be linearly independent.

(2) (Short answer questions.)

(a) Give an example of an inconsistent system. (Explain how you know it is inconsistent.)

The simplest example is probably the 1×1 system: $0 \cdot x = 1$. For any choice of x, the left hand side is 0, which is not 1, so the system is inconsistent.
For a more "linear algebra-like" explanation of inconsistency, one could write down the augmented matrix $[0|1]$ and observe that it is already in RRE form and there is a pivot right of the vertical bar.

(b) Suppose that $\mathbb{V}$ is a 1-dimensional space. Which subsets of $\mathbb{V}$ are bases for $\mathbb{V}$?

The bases for V are exactly the sets $\{\mathbf{v}\}$ which contain one vector $\mathbf{v} \neq \mathbf{0}$.

(3) Let S be the set of all vectors in $\mathbb{R}^2$ whose entries sum to zero. (This is the set of real column vectors of length 2 whose entries a, b satisfy $a + b = 0$.) Does S span $\mathbb{R}^2$? Explain.

No. The set S is closed under scaling and addition, so $\text{span}(S) = S$. S does not contain $\mathbf{e}_1$, so $\text{span}(S) = S \neq \mathbb{R}^2$.

(4) Use linear algebra to find a curve of the form $y = ax^2 + bx + c$ (a parabola) that passes through the points $(x, y) = (-1, 8), (0, 1)$ and $(1, -2)$.

Substituting points yields a system of linear equations ($a - b + c = 8, c = 1, a + b + c = -2$) in the unknowns a, b, c that has the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -2 \end{array}\right].$$

The solution is $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$, which yields $y = 2x^2 - 5x + 1$.

(5) Let A be a square matrix over the real numbers. Explain why, if the columns of A are independent over the real numbers, then the columns of A^2 are independent over the real numbers.

The columns of A are linearly independent iff A has a left inverse, say L. (This means that $LA = I$.) Now it is easy to see that L^2 will be a left inverse for A^2

$$L^2A^2 = L(LA)A = LIA = LA = I,$$

so the columns of A^2 are linearly independent.