LINEAR ALGEBRA (Math 2135): REVIEW SHEET 2

From the book: Chapter 3 (Maps between spaces), Sections I-V; Chapter 4 (Determinants), Sections 1-III; Chapter 5 (Similarity), Section II and IV plus Topics Method of Powers, Stable Populations, Linear Recurrences.

VIII. Linear transformations.

(a) Vector spaces are compared by homomorphisms. A homomorphism between vector spaces is usually called a "linear transformation".
(b) An invertible linear transformation is an isomorphism. To test if a linear transformation is invertible, it is enough to test that it is bijective.
(c) Vector spaces are classified up to isomorphism by dimension.
(d) Any finitely generated $\mathbb{F}$-vector space is isomorphic to $\mathbb{F}^n$ for some finite n.
(e) Universal Mapping Property: any linear transformation is freely and uniquely specified by its action on a basis.
(f) Matrices, ${}_{\mathcal{C}}[T]_{\mathcal{B}}$, for linear transformations between abstract vector spaces. Algorithm to compute change of basis matrices, ${}_{\mathcal{C}}[\mathrm{id}]_{\mathcal{B}}$.
(g) Canonical factorization of a linear transformation: $T = \iota \circ \overline{T} \circ \nu$.

IX. Subspaces.

(a) Algorithm to compute bases for $\mathrm{im}(T)$ and $\ker(T)$ when $T\colon \mathbb{V} \to \mathbb{W}$ is linear and a basis for $\mathbb{V}$ is known.
(b) The lattice of subspaces of $\mathbb{V}$ is complemented. Algorithm to compute a basis for a complement to $S \leq \mathbb{V}$ when bases for S and $\mathbb{V}$ are given.
(c) If $R, S \leq \mathbb{V}$, then $\dim(R+S) = \dim(R) + \dim(S) - \dim(R \cap S)$.

IX. The determinant.

(a) Signed volume.
(b) The determinant is the unique alternating, multilinear function d of n variables defined on $\mathbb{R}^n$ for which $d(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.
(c) Sign of a permutation. Permutation matrices. Permutation expansion.
(d) $\det(A) = \det(A^t)$.
(e) Minor, cofactor, definition of the determinant via the Laplace expansion.
(f) $\det(A)$ is defined only if A is square. $\det(A) \neq 0$ iff the columns of A are independent iff A is invertible.
(g) Adjugate matrix. Fact that $A \cdot \mathrm{adj}(A) = \det(A) \cdot I$, hence $A^{-1} = (1/\det(A))\mathrm{adj}(A)$ when A is invertible.
(h) Further properties: $\det(AB) = \det(A)\det(B)$, the determinant can be computed by Gaussian elimination, the determinant of a block triangular matrix is the product of the determinants of the blocks, if $T(\mathbf{x}) = A\mathbf{x}$, then the determinant of A measures the "volume expansion" associated with T.

X. Eigenvalues, eigenvectors, eigenspaces.
   (a) Eigenvectors identify "preserved directions" or "axes" of a linear endomorphism $T\colon V \to V$.
   (b) Definitions of eigenvector, eigenvalue, eigenspaces.
   (c) Methods of calculation: characteristic polynomial $\chi_A(\lambda)$ equals $\det(\lambda I - A)$; e-values of A are the roots of $\chi_A(\lambda) = 0$; e-space V_λ equals $\mathrm{null}(\lambda I - A) = \mathrm{null}(A - \lambda I)$; λ-eigenvectors are the nonzero vectors of V_λ. Fast calculation of e-values for (block) triangular matrices.
   (d) $\chi_A(\lambda) = \chi_{A^t}(\lambda)$.
   (e) $\mathsf{tr}(A)$ equals the sum of the e-values of A; $\mathsf{det}(A)$ equals the product of the e-values of A.

XI. Diagonalization.
   (a) Every field can be extended to an algebraically closed field (ACF). Over an ACF, every polynomial $p(x)$ in one variable with leading coefficient 1 and degree > 0 factors completely into linear factors: $p(x) = \prod(x - r_i)^{m_i}$, where each r_i is a **root** and m_i is the **multiplicity** of r_i as a root.
   (b) Algebraic multiplicity of an e-value. Geometric multiplicity of an e-value.
   (c) Definition of "diagonalizable". Thm. A transformation $T\colon V \to V$ is diagonalizable iff V has a basis consisting of e-vectors for T iff the geometric multiplicity of each e-value equals its algebraic multiplicity.
   (d) Raising matrices to powers. Solving first-order recurrence relations in many variables through diagonalization.
   (e) Independence of subspaces. A sum of e-spaces for distinct e-values is an independent sum. $T\colon V \to V$ is diagonalizable iff $V = \bigoplus_\lambda V_\lambda$.
   (f) Similarity: B is similar to A (written $B \sim A$) if B is a conjugate of A, i.e., $B = C^{-1}AC$ for some invertible C. Similarity is an equivalence relation on the set of $n \times n$ matrices. Matrices are similar iff they represent the same transformation relative to different bases. Similar matrices have the same characteristic polynomial, hence same e-values each to the same multiplicity. If $B = C^{-1}AC$, then $C^{-1}\colon V_\lambda^A \to V_\lambda^B$ is an isomorphism for each e-value λ. A is diagonalizable iff it is similar to a diagonal matrix.
   (g) Cayley-Hamilton Theorem. Minimal polynomial of a matrix. Relationship between the factorizations of $\chi_A(\lambda)$ and $\mathsf{minpoly}_{A,\mathbb{F}}(\lambda)$. A matrix is diagonalizable over $\mathbb{F}$ iff its minimal polynomial factors into distinct linear factors over $\mathbb{F}$.
   (h) Generalized e-spaces. Eigenchains. Jordan form of a matrix.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.

(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Practice Problems.

(1) Define the following terms.
 (a) Vector space.
 (b) Linear transformation.
 (c) Kernel of a linear transformation.
 (d) Image of a linear transformation.
 (e) Matrix of a transformation relative to an ordered basis.
 (f) Upper triangular matrix.
 (g) Block diagonal matrix.
 (h) Permutation matrix.
 (i) Null space of a matrix.
 (j) Column space of a matrix.
 (k) Determinant of a matrix.
 (l) Characteristic polynomial of a matrix.
 (m) Eigenvector of a linear transformation (or matrix).
 (n) Minimal polynomial of a linear transformation (or matrix).
 (o) Similar matrices.
 (p) Conjugate matrices.
 (q) Diagonalizable matrix.
 (r) Jordan block.
 (s) Nilpotent matrix.

(2) Can any of the following exist? (If so, give an example, if not give a reason.)
 (a) An empty vector space.
 (b) A vector space isomorphic to one of its proper subspaces.
 (c) A linear transformation whose kernel is empty.
 (d) A matrix whose set of columns is independent, but its set of rows is dependent.
 (e) A matrix whose row space is isomorphic to its column space.
 (f) A matrix A satisfying $\mathrm{rank}(A) < \mathrm{rank}(A^2)$.
 (g) An invertible matrix with determinant zero.
 (h) A matrix whose characteristic polynomial is $\lambda^2 + \lambda + 1$.
 (i) A matrix whose eigenvalues are not on the diagonal.
 (j) An eigenvalue whose geometric multiplicity exceeds its algebraic multiplicity.

(k) An invertible matrix with a eigenvalue equal to zero.
(l) A real 10×10 matrix with only one eigenvector.
(m) Conjugate matrices of different ranks.
(n) Conjugate matrices that are not similar.
(o) A real matrix that is diagonalizable over $\mathbb{C}$ but not diagonalizable over $\mathbb{R}$.
(p) A nondiagonalizable complex matrix.
(q) A matrix whose minimal polynomial is a constant polynomial.
(r) A matrix whose minimal polynomial has a repeated root.
(s) A matrix that is both diagonal and nilpotent.
(t) Two nilpotent matrices whose sum is not nilpotent.
(u) Two nilpotent matrices whose product is not nilpotent.

(3) Find a change of basis matrix from $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$ to $\mathcal{C} = \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$.

(4) Find the matrix that rotates $\mathbb{R}^2$ counterclockwise about the origin through an angle of $30°$. Find the matrix that reflects $\mathbb{R}^2$ through the line of positive slope through the origin which makes an angle of $30°$ with the x-axis.

(5) Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the endomorphism $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} b+c+d & a+c+d \\ a+b+d & a+b+c \end{bmatrix}$, which replaces each entry of a 2×2 matrix with the sum of the other entries. What are the e-values and e-spaces of T? Does the space have a basis of e-vectors of T?

(6) How would you solve the following problem? Suppose that V has basis $(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and U is a subspace of V with basis $(\mathbf{u}_1, \ldots, \mathbf{u}_m)$. How do you find a basis for V whose first m vectors form a basis for U?

(7) Suppose that you are given bases $\mathcal{B}$ and $\mathcal{C}$ for subspaces U and W of a space V. How would you find a basis for $U + W$? How would you find a basis for $U \cap W$? (Hint: in both cases, you should apply Gaussian Elimination to the matrix $[\mathcal{B}|\mathcal{C}]$. How should you use the results?)

(8) Find the determinant of $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix}$ in each of the following ways:
(a) The permutation expansion.
(b) The Laplace expansion.
(c) Gaussian elimination.

(9) Let S be a 2×2 invertible real matrix. Consider the linear transformation of "conjugation by S":

$$T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})\colon A \mapsto S^{-1}AS.$$

Show that if $S\mathbf{u} = \lambda\mathbf{u}$, $\mathbf{u} \neq 0$, and $S^t\mathbf{v} = \mu\mathbf{v}$, $\mathbf{v} \neq 0$, then $A = \mathbf{u}\mathbf{v}^t \neq 0$ has the property that $T(A) = (\mu/\lambda)A$. Do you have any conjectures about the set of e-values of T?

(10) This problem involves the $n \times n$ matrix A whose entries are all 1's.
   (a) Without trying to calculate χ_A, determine the e-values of A and their multiplicities.
   (b) Without trying to calculate χ_A, determine the eigenspaces.
   (c) Use the earlier parts of this problem to write down $\chi_A(\lambda)$.
   (d) What is the minimal polynomial of A?

(11) Let A be an $n \times n$ right stochastic matrix. (The entries of A are nonnegative and the row sums are 1.) Show that $|\lambda| \leq 1$ for every e-value for A, and that $\lambda = 1$ for at least one e-value. (Hint for the first part: Suppose that λ is e-value and that v is a λ e-vector. Suppose that $a \neq 0$ is an entry in v with largest absolute value, and that b is an entry in Av with largest absolute value. Show that $|b| \leq |a|$. Deduce that $|\lambda a| \leq |a|$, and then that $|\lambda| \leq 1$.) What if A was left stochastic instead of right stochastic?

(12) Solve the recurrence $a_{n+1} = a_n - a_{n-1} + a_{n-2}$, $a_0 = 0, a_1 = 1, a_2 = 2$.

(13) Suppose that I tell you the e-values, e-spaces, and characteristic polynomial of A. Can you easily describe the e-values, e-spaces, and characteristic polynomial of $A - rI$?

(14) Suppose that I tell you the e-values, e-spaces, and characteristic polynomial of A. Can you easily describe the e-values, e-spaces, and characteristic polynomial of $C^{-1}AC$?

(15) State the Cayley-Hamilton Theorem.