LINEAR ALGEBRA (MATH 2135): REVIEW SHEET

From the book: Chapter 1 (Linear Systems), Sections I, II, III; Chapter 2 (Vector Spaces): Sections I, II, III (+ Topic: Fields).

I. Systems of linear equations.
  (a) Augmented matrix and coefficient matrix of a system.
  (b) Row reduction. (Reduced) row echelon form. Pivot positions, pivot columns.
  (c) Free and basic(=pivot) variables. Solutions sets. Parametrized form of a solution.
  (d) Consistent and inconsistent systems.
  (e) Homogeneous systems. Relationship between solutions of $A\mathbf{x} = \mathbf{b}$ and solutions of $A\mathbf{x} = \mathbf{0}$.

II. Matrix arithmetic.
  (a) Matrices can be added, negated, multiplied with each other, and scaled, provided the dimensions are right.
  (b) Multiplication of $n \times n$ matrices is not commutative if $n > 1$.
  (c) Left and right inverses. (Equivalent properties.) Two-sided inverses. A 1-sided invertible matrix is 2-sided invertible iff it is square.
  The following are equivalent to the left invertibility of A:
    (i) The set of columns of A is independent.
    (ii) Every column of A contains a pivot.
    (iii) $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$.
  The following are equivalent to the right invertibility of A:
    (i) The set of columns of A spans $\mathbb{R}^m$.
    (ii) Every row of A contains a pivot.
    (iii) $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.
  (e) Algorithms for finding left or right inverses.

III. Vectors and vector spaces.
  (a) Linear systems may be viewed as vector equations.
  (b) Definition of vector space.
  (c) Geometric interpretation of vector space operations in $\mathbb{R}^m$.
  (d) Spanning set of vectors.
  (e) Linearly (in)dependent set of vectors.
  (f) Definition of basis. Standard basis for $\mathbb{R}^m$.
  (g) Definition of dimension.
  (h) Definition of subspace. Dimension of a subspace.

IV. Linear Transformations.
  (a) Definition. Fact that any linear transformation has the form $T(\mathbf{x}) = A\mathbf{x}$.
  (b) The problem of solving the linear system $A\mathbf{x} = \mathbf{b}$ may be viewed as the problem of finding a vector $\mathbf{x} \in T^{-1}(\mathbf{b})$ for $T(\mathbf{x}) = A\mathbf{x}$.
  (c) One-to-one and onto transformations.
  (d) Finding the standard matrix of a transformation.
  (e) Matrices for rotation and reflection in the plane.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class.
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
(vi) Know how to correct mistakes made on old HW.

Sample Problems.

(1) Find the equation of a circle in the plane that passes through the points $(-3, 1), (-1, 5), (5, 5)$. (Hint: set up a linear system.)

(2) Let A be a square matrix. Explain why if the columns of A are independent, then the columns of A^2 are independent.

(3) Show that if $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is independent, then $\{\mathbf{a}, \mathbf{a} + \mathbf{b}, \mathbf{a} + \mathbf{b} + \mathbf{c}\}$ is also independent.

(4) Explain why if A and B are $n \times n$ matrices satisfying $A\mathbf{x} = B\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.

(5) Explain why if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a subset of $\mathbb{R}^m$ that is linearly independent and spans the space, then $k = m$.

(6) Matrices A and B commute if $AB = BA$. Which matrices commute with $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$? (Hint: set up a linear system.)

(7) Explain why the set of columns of an $n \times n$ invertible matrix spans $\mathbb{R}^n$. Then explain why this set of columns is independent.

(8) An $n \times n$ matrix M is symmetric if $M^T = M$, and is antisymmetric if $M^T = -M$.
  (a) Show that $S = \frac{1}{2}\left(M + M^T\right)$ is symmetric, $A = \frac{1}{2}\left(M - M^T\right)$ is antisymmetric, and $M = S + A$.
  (b) Show that there is exactly one way to write a square matrix M as $S + A$ with S symmetric and A antisymmetric.

(9) Can any of the following exist? (If so, give an example, if not give a reason.)
  (a) A vector space with an empty basis.
  (b) An invertible matrix whose row sums are all zero.
  (c) A vector space with no subspaces.