Linear Algebra
Quiz 7
Solutions in blue. Remarks in purple.

You have 15 minutes to complete this quiz. In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show your work, and give adequate explanations. You may not give or receive aid on this quiz, and you may not use books, internet, notes, people, etc, (use only your knowledge).

1. Let P_2 be the real vector space of polynomials of degree at most 2. Let $D\colon P_2 \to P_2\colon p(x) \mapsto p'(x)$ be differentiation. Describe the kernel of D.

From Calculus, we know that the kernel of D is the space of constant polynomials.

[Reminder! The **kernel** of a linear transformation is the subspace of the domain consisting of all vectors the transformation maps to zero. Here it is the space of all $p(x) \in P_2$ such that $p'(x) = 0$.]

2. This is a continuation of Question 1. Describe a subspace of P_2 that is complementary to the kernel of D.

$\mathcal{B} = (1, x, x^2)$ is an ordered basis for P_2 and the initial part of this sequence, (1), is an ordered basis for $\ker(D)$. Hence, the sequence (x, x^2) is an ordered basis for a complement of $\ker(D)$. This shows that $\operatorname{span}(x, x^2)$ is a complement to $\ker(D)$.

[Reminder! If S is a subspace of a vector space $\mathbb{V}$, then a **complement** to S is another subspace S' such that (i) $S \cap S' = \{0\}$ (we say that S' and S are **disjoint**), and (ii) $S + S' = \mathbb{V}$ (we say that S' **supplements** S).]

3. (Bonus question!) This is a continuation of Questions 1 and 2. Describe a second subspace of P_2 that is complementary to the kernel of D. (This subspace should be different from the one in your answer to Question 2.)

The complement described in Problem 2 is $\operatorname{span}(x, x^2) = \{ax^2 + bx \mid a, b \in \mathbb{R}\}$. This is the set of all polynomials $p(x) \in P_2$ with zero constant term, which are the polynomials satisfying $p(0) = 0$. Another complement is the set of polynomials $p(x) \in P_2$ such that $p(1) = 0$. This is $\operatorname{span}(x - 1, (x - 1)^2)$.

Remark! It turns out that a subspace of P_2 is a complement of $\ker(D)$ if and only if it has the form $\operatorname{span}(x - r, (x - r)^2)$ for some $r \in \mathbb{R}$.