Solutions to HW 9.

1. Show that if λ is an e-value for A and k is an integer, then λ^k is an e-value for A^k.

The problem should indicate that k is a positive integer, since the statement is false for $\lambda = 0$ if $k < 0$.

Solution.
We prove this by induction on k. Throughout, assume that v is a λ e-vector for A.
Base Case. $k = 1$
In this case, $A^k v = Av = \lambda v = \lambda^k v$.
Inductive step. Assume the statement is true for $k = n$. Let's prove it for $k = n + 1$.
$A^k v = A^{n+1} v = A(A^n v)v = A(\lambda^n v) = \lambda^n Av = \lambda^n \cdot \lambda v = \lambda^{n+1} v.$

2. Suppose that A is a 2×2-matrix and that $\det(A-I) = -16$ and $\det(A-2I) = -15$. What are the e-values of A?

Solution. If X is a 2×2 matrix, then $\mathsf{det}(X) = \mathsf{det}(-X)$. Hence $\chi_A(\lambda) = \mathsf{det}(\lambda I - A) = \mathsf{det}(A - \lambda I)$, and this polynomial is $\lambda^2 - \mathsf{tr}(A)\lambda + \mathsf{det}(A)$. We are given that $\mathsf{det}(A - I) = -16$ so $\chi_A(1) = -16$. Similarly, from $\mathsf{det}(A - 2I) = -15$ we derive that $\chi_A(2) = -15$. Substituting these values into $\lambda^2 - \mathsf{tr}(A)\lambda + \mathsf{det}(A)$ yields

$$\begin{aligned} 1 - \mathsf{tr}(A) + \mathsf{det}(A) &= -16 \\ 4 - 2\mathsf{tr}(A) + \mathsf{det}(A) &= -15 \end{aligned} \quad \text{or} \quad \begin{aligned} -\mathsf{tr}(A) + \mathsf{det}(A) &= -17 \\ -2\mathsf{tr}(A) + \mathsf{det}(A) &= -19 \end{aligned}$$

or

$$\begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \mathsf{tr}(A) \\ \mathsf{det}(A) \end{bmatrix} = \begin{bmatrix} -17 \\ -19 \end{bmatrix}$$

The solution to this system is $\mathsf{tr}(A) = 2$ and $\mathsf{det}(A) = -15$, so $\chi_A(\lambda) = \lambda^2 - 2\lambda - 15$. The roots are $\lambda_1 = -3$ and $\lambda_2 = 5$.

3. Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})\colon M \mapsto M^t$ be the operation of transpose. Find the characteristic polynomial, e-values, and e-spaces of T.

Solution. Choose the basis $\mathcal{B} = (E_{11}, E_{12}, E_{21}, E_{22})$ for $M_{2\times 2}(\mathbb{R})$. With respect to this basis, the matrix for T is

$${}_{\mathcal{B}}[T]_{\mathcal{B}} = A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We compute $\chi_A(\lambda)$ by evaluating a determinant (I used the Laplace expansion along whichever row had the most zeros):

$$\chi_A(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & -1 & \lambda & 0 \\ 0 & 0 & 0 & \lambda - 1 \end{bmatrix} = (\lambda - 1)^2(\lambda^2 - 1) = (\lambda - 1)^3(\lambda + 1),$$

so $\chi_A(\lambda) = (\lambda-1)^3(\lambda+1)$. The e-values of T (or A) are the roots of this polynomial, so they are $\lambda_1 = +1$ (algebraic multiplicity $= 3$) and $\lambda_2 = -1$ (algebraic multiplicity $= 1$).

$V_1 = \text{null}(A-I) = \ker(T-I) = \{M \in M_{2\times 2}(\mathbb{R}) \mid T(M) = M\} = \{M \in M_{2\times 2}(\mathbb{R}) \mid M^t = M\}$, which is the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of symmetric 2×2 matrices. A basis for this subspace is $\mathcal{S} = (E_{11}, E_{12}+E_{21}, E_{22}) = \left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)$. Written in the $\mathcal{B}$-basis, this is

$$\mathcal{S} = \left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}_{\mathcal{B}}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}_{\mathcal{B}}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}_{\mathcal{B}}\right).$$

One way to discover this basis is to apply the nullspace algorithm to

$$A - I = \begin{bmatrix}0 & 0 & 0 & 0\\0 & -1 & 1 & 0\\0 & 1 & -1 & 0\\0 & 0 & 0 & 0\end{bmatrix}.$$

Similarly, $V_2 = \text{null}(A+I) = \ker(T+I) = \{M \in M_{2\times 2}(\mathbb{R}) \mid T(M) = M^t = -M\}$, which is the subspace of $M_{2\times 2}(\mathbb{R})$ consisting of antisymmetric 2×2 matrices. A basis for this subspace is $\mathcal{A} = (E_{12} - E_{21})$. Written in the $\mathcal{B}$-basis, this is

$$\mathcal{A} = \left(\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}_{\mathcal{B}}\right).$$