Solutions to HW 8.

1. Compute the determinant of the following matrix using each of the methods indicated.

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

   (a) The permutation expansion.
   (b) The Laplace expansion along the second column.
   (c) Gaussian elimination.

Solution.
Part (a): There are three permutation matrices in the support of A (that is, whose nonzero entries occur in positions where A has nonzero entries). Using these, we find that

$$\begin{aligned} \det(A) &= 1 \cdot 1 \cdot 3 \cdot \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 1 \cdot 2 \cdot 1 \cdot \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + 2 \cdot 1 \cdot 1 \cdot \det \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ &= 3 - 2 - 2 \\ &= -1. \end{aligned}$$

Part (b):

$$\begin{aligned} \det(A) &= -0 \cdot \det \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \\ &= 0 + 1 - 2 \\ &= -1. \end{aligned}$$

Part (c): Use G.E. without scaling, and keeping track of the number of row interchanges.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[-R_2+R_3]{-R_1+R_3} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

Since there were no row interchanges, the answer is the product of the diagonal entries: $1 \cdot 1 \cdot (-1) = -1$.

2. Explain why the determinant of an upper triangular matrix equals the product of the diagonal entries.

Solution. The only permutation matrix that could be in the support of an upper triangular matrix U is the identity matrix, so according to the permutation expansion of the determinant we get $\det(U) = u_{11} \cdots u_{nn} \cdot \det(I) =$ the product of the diagonal entries of U.

3. Let A and B be $n \times n$ matrices with $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$.

(a) Use the fact that $AB = [A\mathbf{b}_1 \cdots A\mathbf{b}_n]$ to show that, as a function of the columns of B, $f(\mathbf{b}_1, \ldots, \mathbf{b}_n) := \det(AB)$ is multilinear and alternating.

(b) We showed that if f is a multilinear and alternating function of n columns, then

$$f(B) = (\text{perm expansion of } \det(B)) \cdot f(\mathbf{e}_1, \ldots, \mathbf{e}_n).$$

Use this and part (a) to conclude that $\det(AB) = \det(B) \cdot \det(A) = \det(A) \cdot \det(B)$.

Solution.

Part (a): A is a fixed $n \times n$ matrix.

First we argue that the function

$$f(\mathbf{b}_1, \ldots, \mathbf{b}_n) = \det(A \cdot [\mathbf{b}_1 \cdots \mathbf{b}_n)]) = \det([A\mathbf{b}_1 \cdots A\mathbf{b}_n])$$

is alternating. If $\mathbf{b}_i = \mathbf{b}_j$ for some $i \neq j$, then $A\mathbf{b}_i = A\mathbf{b}_j$, so

$$f(\mathbf{b}_1, \ldots, \mathbf{b}_n) = \det([A\mathbf{b}_1 \cdots A\mathbf{b}_n]) = 0$$

since the determinant is alternating.

Next we argue that f is multilinear. In fact, we argue more generally that if $F(x_1, \ldots, x_n)$ is any multilinear function and $T_1(x_1), \ldots, T_n(x_n)$ are linear functions, then the composition $G(x_1, \ldots, x_n) := F(T_1(x_1), \ldots, T_n(x_n))$ is a multilinear function. This more general observation can be used here with $F = \det$ and $T_i(x) = Ax$ for all i.

$$\begin{aligned}
G(\alpha y + \beta z, x_2, \ldots, x_n) &= F(T_1(\alpha y + \beta z), T_2(x_2), \ldots, T_n(x_n)) && \text{Defn of } G \\
&= F(\alpha T_1(y) + \beta T_1(z), T_2(x_2), \ldots, T_n(x_n)) && T_1 \text{ is linear} \\
&= \alpha F(T_1(y), T_2(x_2), \ldots, T_n(x_n)) + \beta F(T_1(z), T_2(x_2), \ldots, T_n(x_n)) && F \text{ linear in 1st var} \\
&= \alpha G(y, x_2, \ldots, x_n) + \beta G(z, x_2, \ldots, x_n) && \text{Defn of } G
\end{aligned}$$

This proves that G is linear in its 1st variable, but the argument works in an other variable, so G is multilinear.

Part (b): following the hint,

$$\det(AB) = f(B) = (\text{perm expansion of } \det(B)) \cdot f(\mathbf{e}_1, \ldots, \mathbf{e}_n) = \det(B) \cdot \det(AI) = \det(B) \cdot \det(A),$$

so, since scalars commute, $\det(AB) = \det(B) \cdot \det(A) = \det(A) \cdot \det(B)$.