Solutions to HW 7.

1. Let $\mathbb{V}$ be the real vector space of $n \times n$ matrices. Let $U \leq \mathbb{V}$ be the subspace of upper triangular matrices and let $L \leq \mathbb{V}$ be the subspace of lower triangular matrices. By computing the necessary dimensions, verify that $\dim(U + L) = \dim(U) + \dim(L) - \dim(U \cap L)$. (Note: a matrix is upper triangular if all entries strictly below the main diagonal are zero, and is lower triangular if all entries strictly above the main diagonal are zero.)

Solution. A typical $n \times n$ upper triangular matrix has the form

$$\begin{bmatrix} a_{11} & a_{11} & \cdots & a_{1n} \\ 0 & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

To specify such a matrix, one must specify one entry from the first column, two from the second, i from the ith, until n from the nth column, so intuitively we guess that the dimension of the space of these matrices is $1+2+\cdots+n = \frac{n(n+1)}{2}$. This intuition can be made rigorous by exhibiting a basis for the space which has this size, namely

$$(\underline{E_{11}}, \underline{E_{12}, E_{22}}, \underline{E_{13}, E_{23}, E_{33}}, \ldots, \underline{E_{1n}, E_{2n}, \ldots, E_{nn}}).$$

This yields $\dim(U) = \frac{n(n+1)}{2}$. A similar calculation shows that $\dim(L) = \frac{n(n+1)}{2}$. Since $U \cap L$ is the space of diagonal matrices, and each diagonal matrix has n independent entries, it is not hard to see that $\dim(U \cap L) = n$. It is also not hard to see that $U + L = \mathbb{V}$, since $U + L$ contains all matrices E_{ij}, and the set of these is a basis for $\mathbb{V}$. In particular, $\dim(U + L) = \dim(\mathbb{V}) = n^2$. (These dimension counts can be worded in a more precise way by establishing that $\{E_{ij} \mid i \leq j\}$ is a basis for U, $\{E_{ij} \mid i \geq j\}$ is a basis for L, $\{E_{ii} \mid 1 \leq i \leq n\}$ is a basis for $U \cap L$, and $\{E_{ij} \mid \text{all } i, j\}$ is a basis for $U + L = \mathbb{V}$.)

To verify that $\dim(U + L) = \dim(U) + \dim(L) - \dim(U \cap L)$ we calculate

$$\dim(U) + \dim(L) - \dim(U \cap L) = \frac{n(n+1)}{2} + \frac{n(n+1)}{2} - n = n^2 = \dim(U + L).$$

2. Let P_3 be the 4-dimensional real vector space of all polynomials in $\mathbb{R}[x]$ that have degree at most 3. Let $S \leq P_3$ be the 2-dimensional subspace of those polynomials $p(x)$ satisfying $p(1) = p(2) = 0$. Find bases for P_3 and S, and a basis for a complement $S^{\perp}$ to S.

Solution. The most obvious basis for P_3 is $\mathcal{B} = (1, x, x^2, x^3)$.

Next we seek a basis for S. A polynomial $p(x)$ belongs to S if it has degree at most 3 and has factors $x - 1$ and $x - 2$, which means that it can written as $p(x) = (x - 1)(x - 2)q(x)$

where $q(x)$ has degree at most 1. This degree restriction on q means that $q(x) = ax + b$ for some a and b. Hence

$$p(x) = (x-1)(x-2)(ax+b) = a\underline{x(x-1)(x-2)} + b\underline{(x-1)(x-2)},$$

which shows that any $p \in S$ is a linear combination of $f(x) = (x-1)(x-2)$ and $g(x) = x(x-1)(x-2)$. Both f and g belong to S, and neither is a scalar multiple of the other, so $\mathcal{C} = (f, g)$ must be a basis for S.

Next we apply the algorithm to thin out a spanning set $(\mathcal{C}|\mathcal{B}) = (f, g, 1, x, x^2, x^3)$ to a basis. (When written in coordinates, this is called the "column space algorithm".) It has the effect of locating a new basis for P_3 which includes f and g.

Since the first four vectors in $(\mathcal{C}|\mathcal{B}) = (f, g, 1, x, x^2, x^3)$ are independent,[1] and P_3 is 4-dimensional, it must be that $(f, g, 1, x)$ is a basis for P_3. The first two are a basis for S, so the second two must span a complement to S.

This shows that one possible answer to the question is $(1, x)$. You might have obtained a different answer, since S has many complements and each one has many bases. The most likely alternative answers for bases for a complement to S are of the form (x^i, x^j), $i \neq j$. Any such pair is a basis for a complement, and these alternatives might arise if you used the vectors in $\mathcal{B}$ in a different order. In general, a correct answer will consist of an independent sequence $(u(x), v(x))$ of length 2, with both $u, v \in P_3$, such that there is no nonzero combination $au(x) + bv(x)$ that has roots at both $x = 1$ and $x = 2$.

3. Use a determinant to find the area of the triangle in $\mathbb{R}^2$ whose vertices are $(1, 1)$, $(3, 61)$, $(101, 3)$. (Hint: a triangle is half a parallelogram.)

Solution. Rigidly translate the triangle so that one vertex is at the origin. You can do this by subtracting $(1, 1)$ from each pair of coordinates: $(0, 0)$, $(2, 60)$, $(100, 2)$. Now the vectors $\begin{bmatrix} 2 \\ 60 \end{bmatrix}$ and $\begin{bmatrix} 100 \\ 2 \end{bmatrix}$ lie along two sides of a parallelogram whose area is twice the desired value. This means the answer is

$$\frac{1}{2} \cdot \det \begin{bmatrix} 100 & 2 \\ 2 & 60 \end{bmatrix} = \frac{1}{2} \cdot (6000 - 4) = 3000 - 2 = 2998.$$

[1] To see this, suppose that $c_1 \cdot f + c_2 \cdot g + c_3 \cdot 1 + c_4 \cdot x = 0$. Then $c_3 + c_4 x = -c_1 f - c_2 g$ is a linear polynomial with roots at $x = 1$ and $x = 2$, so $c_3 + c_4 x = 0 = -c_1 f - c_2 g$. From this and the independence of both $\{1, x\}$ and $\{f, g\}$ we derive that $c_1 = c_2 = c_3 = c_4 = 0$.