Solutions to HW 6.

1. Let A be the 4×4 matrix whose entries are all 1's, and let $T(x) = Ax$.
   (a) Find a basis for the image of T.
   (b) Find a basis for the kernel of T.
   (c) What is the rank of T? (Recall that $\operatorname{rank}(T) = \dim(\operatorname{im}(T))$.)

Solution. The first step is to put A in RRE form:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Part (a): There is a unique pivot column in the RRE form, the first column, so the first column of the original matrix A constitutes a 1-vector basis for the image (=column space). I write $\mathcal{I} = \left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right)$ for this basis.

Part (b): From the RRE form obtained in Part (a), we find that the free variables are x_2, x_3, x_4, and the general solution to the homogeneous system in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

According to our algorithm, $\mathcal{K} = \left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$ is a basis for the kernel.

Part (c): The rank of T is the dimension of the image of T, which is the size of $\mathcal{I}$, namely it is 1.

2. Let A be the matrix for differentiation $D\colon P_3 \to P_3\colon f(x) \mapsto f'(x)$ relative to the ordered basis $(1, x, x^2, x^3)$, and let $T(x) = Ax$.
   (a) Find a basis for the image of T.
   (b) Find a basis for the kernel of T.
   (c) What is the rank of T?

Solution. We know the matrix for A, from HW5(1), so let's put it in RRE form:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Part (a): The last three columns are the pivot columns, so a basis for the image consists of the last three columns of the original matrix. I write $\mathcal{I} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \right)$ for this basis.

Part (b): From the RRE form obtained in Part (a), we find that the only free variable is x_1, and the general solution to the homogeneous system in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

According to our algorithm, $\mathcal{K} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$ is a basis for the kernel.

Part (c): The rank of T is the dimension of the image of T, which is the size of $\mathcal{I}$, namely it is 3.

3. If A is an $n \times n$ matrix, then any matrix of the form $B = C^{-1}AC$ is called a conjugate of A. Explain why the following statement is true: If $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation whose matrix relative to the standard basis $\mathcal{E}$ is $A = {}_{\mathcal{E}}[T]_{\mathcal{E}}$, then the matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ for T relative to some other basis $\mathcal{B}$ is a conjugate of the matrix A.

Solution. Let $A = {}_{\mathcal{E}}[T]_{\mathcal{E}}$, $B = {}_{\mathcal{B}}[T]_{\mathcal{B}}$, $C = {}_{\mathcal{E}}[\mathrm{id}]_{\mathcal{B}}$, and $D = {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{E}}$. It follows from HW5(3) that $CD = {}_{\mathcal{E}}[\mathrm{id}]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{E}} = {}_{\mathcal{E}}[\mathrm{id}]_{\mathcal{E}} = I$ and $DC = {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{E}} \cdot {}_{\mathcal{E}}[\mathrm{id}]_{\mathcal{B}} = {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}} = I$, so $C^{-1} = D = {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{E}}$. Hence $C^{-1}AC = {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{E}} \cdot {}_{\mathcal{E}}[T]_{\mathcal{E}} \cdot {}_{\mathcal{E}}[\mathrm{id}]_{\mathcal{B}} = {}_{\mathcal{B}}[\mathrm{id} \circ T \circ \mathrm{id}]_{\mathcal{B}} = {}_{\mathcal{B}}[T]_{\mathcal{B}} = B$.