Solutions to HW 5.

1. Let P_n be the $\mathbb{R}$-vector space of polynomials of degree at most n. Let $\mathcal{B} = (1, x, \dots, x^n)$ be an ordered basis for this space. Let $D\colon P_n \to P_n\colon f(x) \mapsto f'(x)$ be the differentiation map (which is linear). Determine ${}_{\mathcal{B}}[D]_{\mathcal{B}}$.

Solution. Since $D(x^k) = k \cdot x^{k-1}$, we get

$$
{}_{\mathcal{B}}[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.
$$

2. Using the notation of Problem 1, let $T_n\colon P_n \to P_n$ be defined by $T_n(f(x)) = f(x+1)$. (T_n is linear.)
   (a) Determine ${}_{\mathcal{B}}[T_3]_{\mathcal{B}}$.
   (b) Predict what ${}_{\mathcal{B}}[T_n]_{\mathcal{B}}$ will look like. (You don't have to verify your guess.)

Solution. Since $T_n(x^k) = (x+1)^k = \sum_{r=0}^{k} \binom{k}{r} x^r$, we get

$$
{}_{\mathcal{B}}[T_3]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad {}_{\mathcal{B}}[T_n]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & \binom{n}{0} \\ 0 & 1 & 2 & 3 & \cdots & \binom{n}{1} \\ 0 & 0 & 1 & 3 & \cdots & \binom{n}{2} \\ 0 & 0 & 0 & 1 & \cdots & \binom{n}{3} \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & \binom{n}{n-1} \\ 0 & 0 & 0 & 0 & \cdots & \binom{n}{n} \end{bmatrix}.
$$

3. Explain why
   (a) ${}_{\mathcal{D}}[T]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[S]_{\mathcal{B}} = {}_{\mathcal{D}}[T \circ S]_{\mathcal{B}}$.
   (b) ${}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}} = I$. (Here "id" refers to the "identity transformation" which is the transformation $\mathrm{id}(x) = x$.)

Solution. For Item (a), it suffices to prove that the matrices on the two sides of the equality symbol agree column by column. For this, it suffices to prove that that both agree on $\mathbf{e}_i = [\mathbf{b}_i]_{\mathcal{B}}$.

$$\begin{aligned}\left({}_{\mathcal{D}}[T]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[S]_{\mathcal{B}}\right)[\mathbf{b}_i]_{\mathcal{B}} &= {}_{\mathcal{D}}[T]_{\mathcal{C}} \cdot [S(\mathbf{b}_i)]_{\mathcal{C}} \\ &= [T(S(\mathbf{b}_i))]_{\mathcal{D}} \\ &= [(T \circ S)(\mathbf{b}_i)]_{\mathcal{D}} \\ &= {}_{\mathcal{D}}[T \circ S]_{\mathcal{B}}[\mathbf{b}_i]_{\mathcal{B}}.\end{aligned}$$

For Item (b), we need to explain why the ith column of ${}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}}$ is $\mathbf{e}_i$. The ith column of ${}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}}$ is

$${}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}} \cdot \mathbf{e}_i = {}_{\mathcal{B}}[\mathrm{id}]_{\mathcal{B}} \cdot [\mathbf{b}_i]_{\mathcal{B}} = [\mathrm{id}(\mathbf{b}_i)]_{\mathcal{B}} = [\mathbf{b}_i]_{\mathcal{B}} = \mathbf{e}_i.$$