Solutions to HW 4.

1. Describe a way to view the set $\mathbb{C}$ of complex numbers as a real vector space. (This means: describe the additive structure and how to scale elements by a real number.)

Solution. According to the usual construction of the complex numbers, each $z \in \mathbb{C}$ may be written as $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$. We must explain how to add these objects and scale them by real numbers. We choose the following definitions. Assume that $z = a + bi, w = c + di \in \mathbb{C}$ and $r \in \mathbb{R}$.

(a) (Addition) $z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$.

(b) (Scaling) $r \cdot z = r \cdot (a + bi) = (ra) + (rb)i$.

That's the end of the problem, but let's say more. It is natural to ask: do these choices for addition and scaling satisfy the definition of a "vector space"? The answer is Yes, but to verify this there are 12 laws to check, so 12 proofs are required. That is too much work for a HW problem, but there is a way to avoid all that work.

Call $z = a + bi$ the "ordinary representation" of the complex number z in terms of the real numbers a and b. Introduce a "column representation" by writing z as $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$. We defined the addition and scaling of complex numbers written in the ordinary representation so that, using the column representation, it agrees with the addition and scaling of elements of $\mathbb{R}^2$. We "already know" that $\mathbb{R}^2$ satisfies the 12 laws defining vector spaces, so the definition we gave must satisfy all 12 laws.

2. Let $\mathbb{R}[x]$ be the real vector space of polynomials in the variable x with real coefficients. Let $S \subseteq \mathbb{R}[x]$ be the subset of polynomials whose roots are real numbers. Determine whether S is a subspace of $\mathbb{R}[x]$. (Say the answer, then give the reason.)

Solution. No, S is not a subspace. S is not closed under addition. To see this, note that $f(x) = x^2 - 2x + 1$ and $g(x) = 2x$ have only real roots, but $(f + g)(x) = x^2 + 1$ has non-real roots.

3. Let $\mathbb{R}[x]$ be the real vector space of polynomials in the variable x with real coefficients. Let $Z \subseteq \mathbb{R}[x]$ be the subset of those polynomials with roots at $x = 0, 1, 2$. Determine whether Z is a subspace of $\mathbb{R}[x]$. (Say the answer, then give the reason.)

Solution. Yes, $Z = \{f(x) \in \mathbb{R}[x] \mid f(0) = f(1) = f(2) = 0\}$ is a subspace of $\mathbb{R}[x]$. To see this, observe that

(a) Z is closed under addition: If $f(x), g(x) \in Z$, then $f(0) = f(1) = f(2) = 0 = g(0) = g(1) = g(2)$. Now $(f + g)(0) = f(0) + g(0) = 0 + 0 = 0$. Similarly, $(f + g)(1) = (f + g)(2) = 0$, so $(f + g)(x) \in Z$.

(b) Z is closed under scaling: Assume that $r \in \mathbb{R}$. If $f(x) \in Z$, then $f(0) = f(1) = f(2) = 0$, so $r \cdot f(0) = r \cdot f(1) = r \cdot f(2) = r \cdot 0 = 0$, so $r \cdot f(x) \in Z$.