Solutions to HW 3.

1. Exercise 2.16, parts (b), (c), (d).

Solutions.

(b) Find the angle between the diagonal of the unit cube in $\mathbb{R}^3$ and one of the axes.

The vector $\mathbf{e}_1$ points along the first coordinate axis, while the vector $\mathbf{d} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ points along the diagonal. The cosine of the angle between them is

$$\cos(\theta) = \frac{\mathbf{d} \cdot \mathbf{e}_1}{\|\mathbf{d}\| \cdot \|\mathbf{e}_1\|} = \frac{1}{\sqrt{3}}.$$

Hence $\theta = \cos^{-1}(1/\sqrt{3})$.

(c) Find the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and one of the axes.

If we repeat the same argument as in part (b), but use $\mathbf{d} = \mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_n$, we get

$$\cos(\theta) = \frac{\mathbf{d} \cdot \mathbf{e}_1}{\|\mathbf{d}\| \cdot \|\mathbf{e}_1\|} = \frac{1}{\sqrt{n}}.$$

Hence $\theta = \cos^{-1}(1/\sqrt{n})$.

(d) What is the limit, as n goes to ∞, of the angle between the diagonal of the unit cube in $\mathbb{R}^n$ and any one of the axes?

If $\theta_n = \cos^{-1}(1/\sqrt{n})$, then $\lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \cos^{-1}(1/\sqrt{n}) = \cos^{-1}(\lim_{n\to\infty} 1/\sqrt{n}) = \cos^{-1}(0) = \pi/2$.

2. Let $\mathbf{e}_i$ be the vector of length n whose ith entry is 1 and whose other entries are 0. The set $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is called the standard basis for $\mathbf{R}^n$. Show that the standard basis is independent.

Solution. If $c_1 \cdot \mathbf{e}_1 + \cdots + c_n \cdot \mathbf{e}_n = \mathbf{0}$, then

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Equating entries, we get $c_1 = c_2 = \cdots = c_n = 0$.

3. Let A be an $m \times n$ matrix and let $\mathbf{e}_i$ be a standard basis vector of length n. Explain why the product $A \cdot \mathbf{e}_i$ equals the ith column of A.

Solution. Write A as $[a_{rs}]$ and $\mathbf{e}_i$ as $[u_{r1}]$. Here $u_{r1} = 0$ if $r \neq i$ and $u_{i1} = 1$.

Since A is $m \times n$ and $\mathbf{e}_i$ is $n \times 1$, the product $A \cdot \mathbf{e}_i$ is defined and is of shape $m \times 1$. That is, it is a column vector of length m. The r-entry of this vector is its $r,1$-entry when considered as a matrix, and this is $\sum_{k=1}^{n} a_{rk} u_{k1}$ according to the definition of matrix multiplication. But $u_{k1} = 0$ if $k \neq i$, and $u_{i1} = 1$, so the sum $\sum_{k=1}^{n} a_{rk} u_{k1}$ reduces to a_{ri}. This shows that $A \cdot \mathbf{e}_i$ is a column vector of length m whose r-entry is a_{ri}. The same statement is true for the ith column of A, so $A \cdot \mathbf{e}_i$ equals the ith column of A.

(X) **Optional Fun Challenge! (0 points!)** An ant starts at a point in the plane and walks in a straight line for 1 unit. He then turns left at a right angle and walks in a straight line for 1/2 unit. He then turns left again at a right angle and walks 1/3 unit. He continues to turn left at right angles and walk 1/4 unit, 1/5 unit, 1/6 unit, ETC. As time progresses, this ant spirals closer and closer to a limiting location. At the limit, how far will the ant be from where he started?

Solution. Assume that the ant starts at the origin. The location of the ant approaches the limiting location

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{4} \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 - \frac{1}{3} + \frac{1}{5} - \cdots \\ \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots \end{bmatrix}.$$

You may remember (from Calculus) that $1 - \frac{1}{3} + \frac{1}{5} - \cdots$ converges to $\frac{\pi}{4}$, and that $1 - \frac{1}{2} + \frac{1}{3} - \cdots$ converges to $\ln(2)$. The first of these is derived from $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$ through integration over $[0, 1]$, and the second is derived from $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ through integration over $[0, 1]$.

From the second series, we derive that $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots = \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \cdots) = \frac{1}{2}(\ln(2))$. Hence the limiting location is

$$\begin{bmatrix} \frac{\pi}{4} \\ \frac{\ln(2)}{2}) \end{bmatrix}.$$

At the limit, the ant will be

$$\sqrt{\left(\frac{\pi}{4}\right)^2 + \left(\frac{\ln(2)}{2}\right)^2}$$

units from where it started. (This is about .8585 units.)