Solutions to HW 2.

1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Show by induction that $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.

Solution.
(Base Case ($n = 1$))
Let $A^1 = A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(Inductive Step (Assume true for $n = k$, show true for $n = k + 1$))
Assume that $A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. Then

$$A^{k+1} = A \cdot A^k = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}.$$

2. I have three matrices: A is $m \times n$, B is $p \times q$, and C is $r \times s$. Suppose I tell you that, because of their dimensions, the triple product ABC is not defined, even though the triple product in every other order is defined (i.e., ACB, BAC, BCA, CAB, CBA are all defined). Explain how you know that I made a mistake.

Solution. If ACB is defined (a product of the form $(m \times n)(r \times s)(p \times q)$ makes sense), then $n = r$ and $s = p$. If BAC is defined (a product of the form $(p \times q)(m \times n)(r \times s)$ makes sense), then $q = m$ and $n = r$. If BCA is defined (a product of the form $(p \times q)(r \times s)(m \times n)$ makes sense), then $q = r$ and $s = m$.
Even without considering the other products, CAB and CBA, we have the information that $n = r = q = m = s = p$. A, B and C are therefore (square) $n \times n$ matrices, and their triple product makes sense in any order, so ABC is defined.

3. 3. This problem is about multiplicative inverses of matrices.
   (a) Find the multiplicative inverse of each type of elementary matrix: P_{ij}, and $E_{ij}(r)$ for the case $i \neq j$ and the case $i = j$.

   $P_{ij}^{-1} = P_{ij}$.
   $E_{ij}(r)^{-1} = E_{ij}(-r)$ when $i \neq j$.
   $E_{ii}(r)^{-1} = E_{ij}(r^{-1})$.

   (b) Show that if A^{-1} is the multiplicative inverse of A and B^{-1} is the multiplicative inverse of B, then $B^{-1}A^{-1}$ is the multiplicative inverse of AB.

To show that $B^{-1}A^{-1}$ is the multiplicative inverse of AB, we must show that the two multiply to the identity in either order.

$$\begin{aligned}(B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B \\ &= B^{-1}IB \\ &= B^{-1}B \\ &= I\end{aligned}$$

and

$$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I\end{aligned}$$

The main point to remember about this exercise is: "The inverse of a product AB is the product of the inverses in the reverse order, $B^{-1}A^{-1}$".