Solutions to HW 1.

1. (Exercise 1.22.) For which values of k are there no solutions, many solutions, or a unique solution to this system?

$$\begin{array}{rrl} x & -y & = 1 \\ 3x & -3y & = k \end{array}$$

Solution. The augmented matrix for this system is

$$\left[\begin{array}{rr|r} 1 & -1 & 1 \\ 3 & -3 & k \end{array}\right].$$

The Reduced Row Echelon Form is

$$\left[\begin{array}{rr|r} 1 & -1 & 1 \\ 3 & -3 & k \end{array}\right] \xrightarrow{-3R_1+R_2} \left[\begin{array}{rr|c} 1 & -1 & 1 \\ 0 & 0 & k-3 \end{array}\right]$$

(i) If there is a nonzero pivot to the right of the vertical bar, then the system is inconsistent (= no solution). This happens if $k-3 \neq 0$, or equivalently if $k \neq 3$. Hence there is no solution if $k \neq 3$.

(ii) If $k = 3$, then the matrix reduces to

$$\left[\begin{array}{rr|r} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

The second column does not have a pivot, so the second variable y is a free variable. We can use the first equation to solve for x in terms of the free variable y: $x - y = 1$, so $x = 1 + y$. The full solution in vector form is

$$\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{c} 1+y \\ y \end{array}\right] = \left[\begin{array}{c} 1 \\ 0 \end{array}\right] + y \left[\begin{array}{c} 1 \\ 1 \end{array}\right].$$

In this case there are infinitely many solutions, one for every choice of y.

(iii) There is no value of k for which the system has a unique solution. (All values of k are accounted for in the previous cases.)

2. (Exercise 1.27.) Find the coefficients a, b, and c so that the graph of $f(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$, and $(2, 3)$.

Solution. To pass through the points we must have $f(1) = a(1)^2 + b(1) + c = 2, f(-1) = a(-1)^2 + b(-1) + c = 6$, and $f(2) = a(2)^2 + b(2) + c = 3$, which means that

$$\begin{array}{rrrl} a & +b & +c & = 2 \\ a & -b & +c & = 6 \\ 4a & +2b & +c & = 3. \end{array}$$

The augmented matrix for this system is

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{array}\right]$$

Apply Gausian Elimination to put this matrix in Row Echelon Form.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 6 \\ 4 & 2 & 1 & 3 \end{array}\right] \xrightarrow[-4R_1+R_3]{-R_1+R_2} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & -2 & -3 & -5 \end{array}\right] \xrightarrow{-R_2+R_3} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{array}\right]$$

We can go a bit further and put the matrix in Reduced Row Echelon Form.

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & 4 \\ 0 & 0 & -3 & -9 \end{array}\right] \xrightarrow[-(1/3)R_3]{-(1/2)R_2} \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[-R_3+R_1]{-R_2+R_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

From the final matrix we can read off $c = 3, b = -2, a = 1$, so $f(x) = x^2 - 2x + 3$.

3. Find all triples (x, y, z) of positive real numbers satisfying the nonlinear system

$$\begin{aligned} x^2y &= 2 \\ xy^2z &= 4 \\ yz^2 &= 8 \end{aligned}$$

using the following method. First, take the logarithm of each equation. Then replace $\log(x)$ with a new symbol X, $\log(y)$ with Y, $\log(z)$ with Z, and solve the resulting linear system for (X, Y, Z). Use your answer to solve the original system. (You can use a log to any base, but it is probably easiest to use base 2.)

Following the hint, we take base 2 logarithm of each equation:

$$\begin{array}{rrrl} 2\log_2(x) & +\log_2(y) & & = 1 \\ \log_2(x) & +2\log_2(y) & +\log_2(z) & = 2 \\ & \log_2(y) & +2\log_2(z) & = 3 \end{array}$$

Replace $\log_2(x)$ with X, $\log_2(y)$ with Y, $\log_2(z)$ with Z to obtain

$$\begin{array}{rrrl} 2X & +Y & & = 1 \\ X & +2Y & +Z & = 2 \\ & Y & +2Z & = 3. \end{array}$$

Solve:

$$\left[\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 3 \end{array}\right] \xrightarrow[-(2/3)R_2+R_3]{-(1/2)R_1+R_2} \left[\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ 0 & 3/2 & 1 & 3/2 \\ 0 & 0 & 4/3 & 2 \end{array}\right] \xrightarrow[(3/4)R_3]{(1/2)R_1,(2/3)R_2} \left[\begin{array}{rrr|r} 1 & 1/2 & 0 & 1/2 \\ 0 & 1 & 2/3 & 1 \\ 0 & 0 & 1 & 3/2 \end{array}\right]$$

We can put the last matrix in RRE Form as follows:

$$\left[\begin{array}{rrr|r} 1 & 1/2 & 0 & 1/2 \\ 0 & 1 & 2/3 & 1 \\ 0 & 0 & 1 & 3/2 \end{array}\right] \xrightarrow[-(1/2)R_2+R_1]{-(2/3)R_3+R_2} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 \end{array}\right].$$

We read off the solution from the final matrix: $Z = 3/2, Y = 0, X = 1/2$. Therefore $\log_2(z) = 3/2, \log_2(y) = 0, \log_2(x) = 1/2$. Therefore $z = 2^{3/2}, y = 2^0 = 1, x = 2^{1/2}$.