**Solutions to HW 10.**

1. What is the minimal polynomial of the $3 \times 3$ matrix of all 1's? Does your answer suggest that the matrix is diagonalizable or nondiagonalizable?

**Solution.** (Assume that $3 \neq 0$ in the scalar field. This happens, for example, when the scalar field is $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, or the 2-element field $\mathbb{F}_2$.)

$\chi_A(\lambda) = \lambda^2(\lambda - 3)$. The minimal polynomial is a monic polynomial that divides $\chi_A$ and has the same roots, so it is either $\lambda^2(\lambda - 3)$ or $\lambda(\lambda - 3)$. We can check that

$$A(A - 3I) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = 0,$$

so, since $A$ satisfies the lesser degree possibility $\lambda(\lambda - 3)$, the minimal polynomial must be $\lambda(\lambda - 3)$.

The fact that the minimal polynomial factors into distinct linear factors over the scalar field implies that $A$ is diagonalizable.

(Assume now that $3 = 0$ in the scalar field. This happens, for example, when the scalar field is $\mathbb{F}_3$.)

Everything works the same way as above, but now $\chi_A(\lambda) = \lambda^2(\lambda - 3) = \lambda^3$, and the minimal polynomial could be $\lambda^3$, $\lambda^2$, or $\lambda$. We can check that $A$ is not a root of the polynomial $\lambda$, since $A \neq 0$, but it is a root of $\lambda^2$, since $A^2 = 3A = 0$. It follows that the minimal polynomial of $A$ is $\lambda^2$. Since the minimal polynomial does not factor into distinct linear factors over this scalar field, $A$ is not diagonalizable.

2. How is the minimal polynomial of A related to the minimal polynomial of $A - I$?

**Solution.** For any matrix $A$ and any monic polynomial $p(x)$, it is the case that $p(A) = 0$ iff $q(A - I) = 0$ for $q(x) = p(x + 1)$. That is, $A$ satisfies a monic polynomial $p(x)$ iff $A - I$ satisfies the same-degree monic polynomial $p(x + 1)$. Similarly $A - I$ satisfies a monic polynomial $q(x)$ iff $A$ satisfies the same-degree monic polynomial $q(x - 1)$. Since the minimal polynomial of a matrix is the monic polynomial of least degree that is satisfied by the matrix, we get that $p(x)$ is the minimal polynomial of $A$ iff $p(x + 1)$ is the minimal polynomial of $A - I$.

3. Show that if $A$ is diagonalizable, then $A - I$ is diagonalizable.

**Solution.** If $C^{-1}AC = \Lambda$ is diagonal, then $C^{-1}(A - I)C = C^{-1}AC - C^{-1}IC = \Lambda - I$ is diagonal.