# Jordan form

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The Jordan form of a matrix is unique up to a permutation of the Jordan blocks. $\square$

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- $\begin{bmatrix} J_2(1) & 0 \\ 0 & J_2(3) \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$

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$$C^{-1}AC = \begin{bmatrix} D_{d_1}(\lambda_{i_1}) & 0 & \cdots & 0 \\ 0 & D_{d_2}(\lambda_{i_2}) & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & D_{d_k}(\lambda_{i_k}) \end{bmatrix} + \begin{bmatrix} N_{d_1} & 0 & \cdots & 0 \\ 0 & N_{d_2} & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & N_{d_k} \end{bmatrix} = D + N$$

where $DN = ND$.

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**Solution 2.**

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- The Jordan form of small-dimension matrices can be determined by $\chi_A$ and rank calculations. (E.g., no ambiguity up to $3 \times 3$.)