

Fields.

Definition 1. A **field** is an algebraic structure $\mathbb{F} = \langle F; +, -, 0, \cdot, 1 \rangle$ which satisfies the following

- (1) Additive laws¹:
 - (a) (Associative law) $\forall x \forall y \forall z ((x + (y + z)) = ((x + y) + z))$.
 - (b) (Commutative law) $\forall x \forall y (x + y = y + x)$.
 - (c) (Unit law) $\forall x (x + 0 = x)$
 - (d) (Inverse law) $\forall x (x + (-x) = 0)$
- (2) Multiplicative laws:
 - (a) (Associative law) $\forall x \forall y \forall z ((x(yz)) = ((xy)z))$.
 - (b) (Commutative law) $\forall x \forall y (xy = yx)$.
 - (c) (Unit law) $\forall x (x1 = x)$
- (3) Law linking addition to multiplication:
 - (a) (Distributive law): $\forall x \forall y \forall z (x(y + z) = xy + xz)$.
- (4) Other defining properties that are not laws:
 - (a) $0 \neq 1$.
 - (b) $\forall x ((x \neq 0) \rightarrow \exists y (xy = 1))$.

If you know the definition of “commutative group”, the axioms for fields say that \mathbb{F} is additively a commutative group, $\mathbb{F} - \{0\}$ is multiplicatively a commutative group, and the additive and multiplicative structures are linked by the distributive law.

These axioms say the following: a field is an algebraic structure that satisfies all equational laws true in the real numbers, and satisfies the properties listed under Item (4) ($0 \neq 1$, and also satisfies the property that every nonzero element has a multiplicative inverse).

Examples. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ and $\mathbb{Z}/p\mathbb{Z}$ (integers modulo p , p prime).

Nonexamples. \mathbb{N}, \mathbb{Z} , and $\mathbb{Z}/n\mathbb{Z}$ (integers modulo n , n not prime).

The smallest field has two elements. It is $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$. Its operation tables are

+	0	1
0	0	1
1	1	0

-	0	1
	0	1

·	0	1
0	0	0
1	0	1

The second smallest field is $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$. Its tables are

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

-	0	1	2
	0	2	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

¹A **law** or **identity** is a universally quantified equation.