Eigenvectors, eigenvalues, characteristic equation, diagonalization

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Theorem. Let A be the matrix for an endomorphism T, written in a basis $\mathcal{B} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$. A is a diagonal matrix iff $\mathcal{B}$ consists of e-vectors for A. In fact,

$$\begin{aligned} A = {}_{\mathcal{B}}[T]_{\mathcal{B}} &= [[T(\mathbf{b}_1)]_{\mathcal{B}} \cdots [T(\mathbf{b}_n)]_{\mathcal{B}}] \\ &= [[\lambda_1 \cdot \mathbf{b}_1]_{\mathcal{B}} \cdots [\lambda_n \cdot \mathbf{b}_n]_{\mathcal{B}}] \\ &= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = \Lambda \end{aligned}$$

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Are we stuck? Not completely. Extend the scalar field $\mathbb{R}$ to the larger (algebraically closed) field $\mathbb{C}$ and work there. E-values are $+i, -i$. Basis of e-vectors is $\mathcal{B} = \left(\begin{bmatrix} i \\ 1 \end{bmatrix}, \begin{bmatrix} -i \\ 1 \end{bmatrix} \right)$. If $C = [\mathcal{B}]$, then $C^{-1}AC = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

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But no diagonal matrix can satisfy both $D \neq \mathbf{0}$ and $D^2 = \mathbf{0}$.

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Remark. If the only obstacle to diagonalizability is that the roots of $\chi_A(\lambda)$ do not belong to $\mathbb{F}$, then extend $\mathbb{F}$ to some larger field (like the algebraic closure $\overline{\mathbb{F}} \supseteq \mathbb{F}$). If, now, $\mathrm{mult}_{\mathsf{geom}}(\lambda) = \mathrm{mult}_{\mathsf{alg}}(\lambda)$ for each root λ of $\chi_A(\lambda)$ in $\overline{\mathbb{F}}$, one can diagonalize A over $\overline{\mathbb{F}}$.

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It follows from the Cayley-Hamilton Theorem that $\mathsf{minpoly}_{A,\mathbb{F}}(\lambda)$ divides $\chi_A(\lambda)$.

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2. Recall that if $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, then $\chi_A(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$. The theorem yields nondiagonalizability over $\mathbb{R}$ but diagonalizability over $\mathbb{C}$.

Some applications

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$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}, \quad \begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}.$$

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$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2}\mathbf{v}_1 - \frac{1}{\lambda_1 - \lambda_2}\mathbf{v}_2$, so $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\lambda_1^k}{\lambda_1 - \lambda_2}\mathbf{v}_1 - \frac{\lambda_2^k}{\lambda_1 - \lambda_2}\mathbf{v}_2$, so

$$F_k = \frac{\lambda_1^k}{\lambda_1 - \lambda_2} - \frac{\lambda_2^k}{\lambda_1 - \lambda_2} = \underbrace{\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k}_{\text{Binet formula}}.$$