

## DISCRETE MATH (MATH 2001)

### REVIEW SHEET II

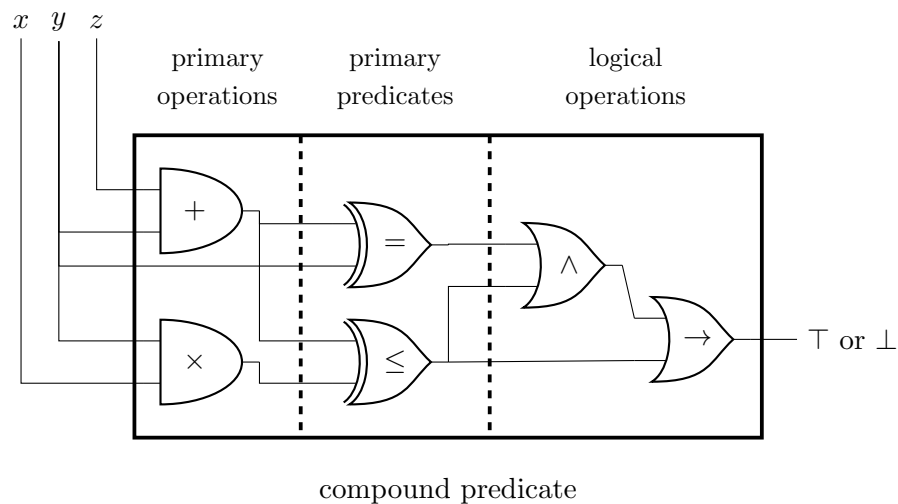
Sections 2.7, 2.8, 3.1, 3.2, 3.5, 3.6.1, 4.1–4.4, 4.6.1, 4.8, 6.1–6.3, 6.5, 6.6

#### IV. Cardinality.

- (a) Ordinal numbers versus cardinal numbers.
- (b) Meaning of  $|A| \leq |B|$ ,  $|A| = |B|$ , and  $|A| < |B|$ .
- (c) Finite and infinite. Countable and uncountable.
- (d) Cantor-Bernstein-Schroeder Theorem.
- (e) Cantor's Theorem.
- (f)  $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = |\mathbb{R}^n|$ .

#### V. Logic.

- (a) Structures.
- (b) Alphabet of symbols. Ingredients in a compound predicate.



- (c) Deciding the truth of a statement in a structure.
  - (i) Assigning tables to terms.
  - (ii) Assigning tables to atomic formulas.
  - (iii) Logical connectives. Truth Tables. Propositional tautology. Contradiction. Logical equivalence of propositions.
  - (iv) Disjunctive normal form.
  - (v) Prenex form.
  - (vi) Quantifier games. Winning strategies.
- (d) Proof.
  - (i) Definition of "proof".
  - (ii) Axioms. Logically valid sentence.
  - (iii) Laws of deduction. Modus ponens, modus tollens.

- (iv) Direct proof, proof of the contrapositive, and proof by contradiction. Proof by cases.
- (v) The use of truth tables for designing proof strategies.
- (vi) The relationship between truth and provability: semantic consequence ( $\Sigma \models S$ ) versus syntactic consequence ( $\Sigma \vdash S$ ).
- (vii) Significance of Soundness, Completeness, and Decidability with regard to proof systems.
- (viii) Relevance of Gödel's Completeness Theorem.

## VI. Counting.

- (a) Additive counting principle and multiplicative counting principle.
- (b) # functions  $f : k \rightarrow n$ . Characteristic functions,  $|\mathcal{P}(n)|$ .
- (c) # injective functions  $f : k \rightarrow n$ .
- (d) # bijective functions  $f : k \rightarrow n$ .
- (e) Binomial coefficients: definition, formula, recursion, Binomial Theorem, Pascal's Triangle.
- (f) Multinomial coefficients: definition, formula, recursion, Multinomial Theorem, Pascal's Pyramid.
- (g) Multichoose numbers: definition, formula.
- (h) Inclusion-exclusion.
- (i) # surjective functions  $f : k \rightarrow n$ .
- (j) Stirling numbers of the second kind.

## General advice on preparing for a math test.

Be prepared to demonstrate understanding in the following ways.

- (i) Know the definitions of new concepts, and the meanings of the definitions.
- (ii) Know the statements and meanings of the major theorems.
- (iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
- (iv) Know how to perform the different kinds of calculations discussed in class.
- (v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)
- (vi) Know how to correct mistakes made on old HW.

## Some definitions to know.

- (1) Finite. Infinite. Countable. Uncountable.  $|A| \leq |B|$ .  $|A| = |B|$ .
- (2) Predicate. Operation. Structure.
- (3) Logical connective. Truth table. Tautology. Contradiction.
- (4) Contrapositive. Converse. Inverse.
- (5) Disjunction. Conjunction. Disjunctive normal form.
- (6) Proof. Axioms. Rules of deduction.
- (7) Valid sentence.
- (8) Semantic consequence. Syntactic consequence.
- (9) Soundness. Completeness.

- (10) Additive counting principal. Multiplicative counting principal.
- (11) Binomial coefficient. Multinomial coefficient. Multichoose number. Stirling number.

**Some theorems to know.**

- (1) Cantor-Bernstein-Schroeder Theorem.
- (2) Cantor's Theorem.
- (3) Gödel's Completeness Theorem.
- (4) Binomial Theorem. Multinomial Theorem.
- (5) Inclusion-Exclusion Theorem.

**Practice Problems.**

- (1) How do you answer a question where you are asked to "Give an example"?

Give an example of such a question.

- (2) How do you answer a question where you are asked to "Give a definition"?

Define "definition".

- (3) If you are asked to "Give a proof or counterexample", how do you decide which thing to do?

Give a proof or counterexample to the claim "Every prime is odd."

- (4) Write a formal sentence expressing the axiom of union. Then draw a formula tree for your sentence.

- (5) A theorem with two hypotheses has the form  $((H_1 \wedge H_2) \rightarrow C)$ . Which of the following implications indicates a valid proof strategy?

- (a)  $(H_1 \wedge (\neg C)) \rightarrow (\neg H_2)$ ?
- (b)  $((\neg C) \rightarrow ((\neg H_1) \wedge (\neg H_2)))$ ?
- (c)  $(H_1 \rightarrow (H_2 \rightarrow C))$ ?

- (6) The contrapositive of the converse of an implication  $H \rightarrow C$  is called the **inverse** of the implication. (I.e.,  $(\neg H) \rightarrow (\neg C)$  is the inverse of  $H \rightarrow C$ .) For example, if the original statement is "If you pay him, then he will be happy", then the inverse statement might be written "If you do not pay him, then he will not be happy". What is the logical relationship between an implication and its inverse? (Answer in one of the following ways: (a) they are logically equivalent, (b) the statement logically implies its inverse, (c) the inverse logically implies the statement, or (d) the statement and its inverse are logically independent.) Explain your answer.

- (7) Describe a winning strategy for the relevant quantifier which determines the truth of

$$\forall x \forall y \exists z ((x < y) \rightarrow ((x < z) \wedge (z < y)))$$

in (a)  $\langle \mathbb{R}; < \rangle$ , (b)  $\langle \mathbb{Z}; < \rangle$ .

- (8) State the following theorems.

- (a) Cantor-Bernstein-Schroeder Theorem.
- (b) Cantor's Theorem.
- (c) Gödel's Completeness Theorem.

- (9) Prove that the empty set is finite.

- (10) How many license plates have exactly 7 characters consisting of decimal digits  $(0, 1, 2, \dots, 9)$  and letters of the alphabet  $(a, b, c, \dots, z)$ ? What if there are exactly 3 decimal digits, 4 letters of the alphabet, and the digits must come before the letters?

- (11) How many license plates have 3 decimal digits and 4 letters of the alphabet if the plate must start and end with a digit, and either start with 9 or end with 9?

- (12) How many ways are there to distribute 12 different books to 3 people? What if each person must get at least one book?

- (13) How many ways are there to distribute 12 identical textbooks to three shelves? How many ways to distribute 12 different books to three shelves?

- (14) How many positive integral solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100?$$

How many nonnegative integral solutions are there?

- (15) How many ways are there to make 3 fruit baskets from 8 pineapples, 10 pomegranates, 6 coconuts and 20 figs if each basket must contain each kind of fruit?

- (16) In your department, everyone works in one of three workgroups. Workgroups A and B each have a total of 23 workers; Workgroup C has a total of 32 workers; there are 5 workers who belong in both groups A and B; 10 workers who belong in both B and C; 8 workers who belong in both A and C; and three workers who belong in all three groups. How many workers are in your department?

- (17) How many positive integers less than 1000 are not divisible by 2, 3, 5 or 7?
- (18) How many 5 digit numbers fail to contain the sequence 01? How about 00?
- (19) How many 6 digit numbers have the property that, for every  $k$ , the  $k$ th digit is different than the  $(7 - k)$ th digit?
- (20) What is the probability that a 3-element multiset of elements from  $\{1, 2, 3, 4, 5\}$  has no repeated elements? (Express your answer as a reduced fraction.)
- (21) Give a combinatorial proof that  $S(n, n - 1) = C(n, 2)$ .
- (22) Suppose that  $|A| = n$  and  $|B| = m$ . How many pairs  $(X, Y)$  are there where  $X = \text{coim}(f)$  and  $Y = \text{im}(f)$  for some function  $f: A \rightarrow B$ ?