Solutions to HW 9.

1. Show that "$A \subseteq B$ and $B \subseteq A$ implies $A = B$" in each of the following two ways.

(i) With a direct proof.

Assume that $A \subseteq B$ and that $B \subseteq A$.

Choose any $a \in A$. Since $A \subseteq B$, it follows that $a \in B$. Now choose any $b \in B$. Since $B \subseteq A$, it follows that $b \in A$. We have shown that A and B have the same elements, so $A = B$ by the Axiom of Extentionality.

(ii) With a proof by contradiction.

Assume that $A \subseteq B$ and that $B \subseteq A$, but $A \neq B$.

Since $A \neq B$, then by the Axiom of Extentionality it is not true that $\forall z((z \in A) \leftrightarrow (z \in B))$, so there is some $z \in A \setminus B$ or some $z \in B \setminus A$. The two cases are similar, so assume that there is some $z \in A \setminus B$. This z witnesses that $A \not\subseteq B$, contrary to our assumption.

2. Show that "any nonconstant, real, linear function $f(x) = ax + b$ has a unique root" in each of the following two ways.

(i) With a direct proof.

Assume that $f(x) = ax + b$ is a nonconstant function.

If $f(x) = ax+b$ is nonconstant, then is must be that $a \neq 0$ (since, when $a = 0$, $f(x) = 0x+b = b$ IS constant). Now, using the arithmetic of the real numbers, $f(x) = 0$ holds exactly when $ax+b = 0$, or $ax = -b$, or $x = -b/a$.

The fact that the equation $ax + b = 0$ is solvable shows that a root exists, the fact that the equation is uniquely solvable shows that the root is unique.

(ii) With a proof by contradiction.

Assume that $f(x) = ax + b$ is nonconstant, but does not have a unique root. If $a = 0$, then $f(x) = ax + b = 0x + b = b$ is constant which is not true, so we may assume that $a \neq 0$.

If $f(x) = ax + b$ does not have a unique root, then either (1) it has no root at all, or (2) it has multiple roots.

Case 1. $f(x) = ax + b$ has no root at all.

By substitution we see that $f(-b/a) = 0$, so f has at least one root when $a \neq 0$. Thus our assumptions place us in

Case 2. f has multiple roots.

If $r_1 \neq r_2$ are roots of f, then $f(r_1) = ar_1 + b = 0$ and $f(r_2) = ar_2 + b = 0$. Subtracting these equations yields $0 = (ar_1 + b) - (ar_2 + b) = a(r_r - r_2)$. Dividing through by $r_1 - r_2$ yields $0 = a$, contrary to an earlier conclusion. This contradiction completes the proof.

3. The goal of this problem is to prove that the composition of two surjective functions is surjective. The type of structure involved looks like $\mathbb{X} = \langle A, B, C; f, g \rangle$ where $f : A \to B$ and $g : B \to C$ are functions. Let the variable a range over the set A, the variable b range over the set B, and the variable c range over the set C.

   The functions (i) f, (ii) g, (iii) $g \circ f$ are surjective if the following sentences hold in $\mathbb{X}$:

   (i) $(\forall b)(\exists a)(f(a) = b)$.

   (ii) $(\forall c)(\exists b)(g(b) = c)$.

   (iii) $(\forall c)(\exists a)(g \ \circ f(a) = c)$.

   To prove that the composition of surjective functions is surjective, you must give a winning strategy for $\exists$ in the sentence in (iii). YOU ARE ALLOWED TO USE the fact that there exist winning strategies for $\exists$ in the sentences in (i) and (ii). Write a proof that indicates the winning strategy for $\exists$ in (iii), which accesses the information of the strategies for $\exists$ in (i) and (ii).

We must provide a winning strategy for $\exists$ for Game (iii). We are allowed to use that there exist winning strategies for $\exists$ in Games (i) and (ii).

How do we access the strategies for $\exists$ in Games (i) and (ii) when developing a strategy for $\exists$ in Game (iii)?

Imagine that Games (i) and (ii) are played by the "earlier self" of $\exists$. When $\exists$ needs information about those games, she will ask her earlier self to *reveal those winning strategies.* She does this by pretending to be $\forall$, and playing Games (i) and (ii) against her earlier self, and watching how her earlier self responds to each move. Then she uses that information while developing a strategy for the current game, Game (iii).

Let's start developing the winning strategy for $\exists$ for Game (iii): $(\forall c)(\exists a)(g \circ f(a) = c)$.

- $\forall$ chooses some $c = c_0 \in C$.
- $\exists$ chooses some $a = a_0 \in A$ using the following strategy:
  - $\exists$ pretends to be the earlier self of $\forall$ in Game (ii), and plays $c = c_0$.
  - the earlier self of $\exists$ in Game (ii) has a winning strategy for Game (ii), so has a way to respond with some $b = b_0 \in B$ such that $g(b_0) = c_0$.
  - Now $\exists$ pretends to be the earlier self of $\forall$ in Game (i), and plays $b = b_0$.
  - the earlier self of $\exists$ in Game (i) has a winning strategy for Game (i), so has a way to respond with some $a = a_0 \in B$ such that $f(a_0) = b_0$.

This choice of $a = a_0 \in A$ is a winning response for $\exists$ against $\forall$'s choice of $c = c_0 \in C$, since if $\exists$ chooses $b = b_0 \in B$ and $a = a_0 \in A$ in these ways, then according to the strategies of Games (i) and (ii) we will have $f(a_0) = b_0$ and $g(b_0) = c_0$, so $c = c_0 = g(f(a_0)) = g \circ f(a)$.