Solutions to HW 5.

Addition

$$\begin{aligned} m+0 &:= m && \text{(Initial Condition for } +) \\ m+S(n) &:= S(m+n) && \text{(Recurrence Relation (or Propagation Rule) for } +) \end{aligned}$$

Multiplication

$$\begin{aligned} m\cdot 0 &:= 0 && \text{(Initial Condition for } \cdot) \\ m\cdot S(n) &:= m\cdot n+m && \text{(Recurrence Relation for } \cdot) \end{aligned}$$

1. Prove that $m(n+k)=(mn)+(mk)$ holds for the natural numbers.

This is a proof by induction on k.
(Base Case: $k=0$)

$$\begin{aligned} m(n+0) &= mn && (\text{IC}, +) \\ &= mn+0 && (\text{IC}, +) \\ &= mn+m0 && (\text{IC}, \cdot) \end{aligned}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned} m(n+S(k)) &= mS(n+k) && (\text{RR}, +) \\ &= m(n+k)+m && (\text{RR}, \cdot) \\ &= (mn+mk)+m && (\text{IH}) \\ &= mn+(mk+m) && (\text{Associative Law}, +) \\ &= mn+mS(k) && (\text{RR}, \cdot) \end{aligned}$$

2. Prove that $m(nk)=(mn)k$ holds for the natural numbers.

This is a proof by induction on k.
(Base Case: $k=0$)

$$\begin{aligned} m(n0) &= m0 && (\text{IC}, \cdot) \\ &= 0 && (\text{IC}, \cdot) \\ &= (mn)0 && (\text{IC}, \cdot) \end{aligned}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned} m(n\cdot S(k)) &= m(nk+n) && (\text{RR}, \cdot) \\ &= m(nk)+mn && (\text{Distributive Law}) \\ &= (mn)k+mn && (\text{IH}) \\ &= (mn)\cdot S(k) && (\text{RR}, \cdot) \end{aligned}$$

3. Prove that $mn = nm$ holds for the natural numbers. (Some lemmas will be needed.)

Lemma 1. $0k = 0$.

Proof. This is a proof by induction on k.
(Base Case: $k = 0$)

$$00 = 0 \qquad (\text{IC}, \cdot)$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned} 0 \cdot S(k) &= 0k + 0 && (\text{RR}, \cdot) \\ &= 0 + 0 && (\text{IH}) \\ &= 0 && (\text{IC}, +) \end{aligned}$$

Lemma 2. (Right distributivity) $(m+n)k = mk + nk$.

Proof. This is a proof by induction on k.
(Base Case: $k = 0$)

$$\begin{aligned} (m+n)0 &= 0 && (\text{IC}, \cdot) \\ &= 0 + 0 && (\text{IC}, +) \\ &= m0 + n0 && (\text{IC}, \cdot) \end{aligned}$$

(Inductive Step: Assume true for k, prove true for $S(k)$)

$$\begin{aligned} (m+n) \cdot S(k) &= (m+n)k + (m+n) && (\text{RR}, \cdot) \\ &= (mk + nk) + (m+n) && (\text{IH}) \\ &= mk + (nk + (m+n)) && (\text{Associative Law}, +) \\ &= mk + ((nk+m)+n)) && (\text{Associative Law}, +) \\ &= mk + ((m+nk)+n)) && (\text{Commutative Law}, +) \\ &= mk + (m + (nk+n)) && (\text{Associative Law}, +) \\ &= (mk+m) + (nk+n) && (\text{Associative Law}, +) \\ &= (m \cdot S(k)) + (n \cdot S(k)) && (\text{RR}, \cdot) \end{aligned}$$

Solution to the Problem.
Proof. We prove that $mn = nm$ by induction on n.
(Base Case: $n = 0$)

$$\begin{aligned} m0 &= 0 && (\text{IC}, \cdot) \\ &= 0m && (\text{Lemma 1}) \end{aligned}$$

(Inductive Step: Assume true for n, prove true for $S(n)$)

$$\begin{aligned} m \cdot S(n) &= mn + m && (\text{RR}, \cdot) \\ &= nm + m && (\text{IH}) \\ &= (n+1)m && (\text{Lemma 2}) \\ &= S(n) \cdot m && (S(n) = n+1) \end{aligned}$$