Solutions to HW 3.

1. Explain why it is true that the function $F : A \to \mathcal{P}(A) : a \mapsto \{a\}$ is injective.

If $F(a) = F(b)$, then $\{a\} = \{b\}$, according to the definition of F. Now, by the Axiom of Extensionality, $a = b$. This establishes that F is injective. (We showed that $F(a) = F(b)$ implies $a = b$.)

2. In this problem, $f : A \to B$ and $g : B \to C$ will be composable functions.

(a) Show that if $g \circ f$ is injective, then f is injective, while if $g \circ f$ is surjective, then g is surjective.

For the first part of the problem, we argue by contradiction. Assume that $g \circ f$ is injective, but that f is not injective. Since f is not injective, there exist distinct $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. But now, since g is a function, $g(f(a_1)) = g(f(a_2))$. This may also be written as $g \circ f(a_1) = g \circ f(a_2)$. By the injectivity of $g \circ f$, we get that $a_1 = a_2$, which contradicts the distinctness of a_1 and a_2.

We give a direct argument for the second part of the statement. We assume that $g \circ f : A \to C$ is surjective, and it is our goal to prove that $g : B \to C$ is surjective. To this end, choose $c \in C$ arbitrarily; our aim is to show that there exists $b \in B$ such that $g(b) = c$. Since $g \circ f : A \to C$ is surjective, there is some $a \in A$ such that $c = g \circ f(a) = g(f(a))$. If we take $b = f(a)$, then we obtain that $g(b) = g(f(a)) = c$, as desired.

(b) Give an example where $g \circ f$ is injective, but g is not injective, and an example where $g \circ f$ is surjective but f is not surjective.

Two examples are requested. I give one example that satisfies both requests.

Let $A = \{0\} = C$ and let $B = \{0, 1\}$. Let $f : A \to B$ be defined so that $f(0) = 0$, and let $g : B \to C$ be defined so that $g(0) = g(1) = 0$. Then $g \circ f : A \to C$ is the identity function, so it is both injective and surjective. But f is not surjective and g is not injective.

3. The function $P_A : A \times B \to A : (a, b) \mapsto a$ is called the first projection map, or the projection onto A.

(a) What is the image of this function? (Make sure to consider the possibility where $B = \emptyset$.)

If $B \neq \emptyset$, then $\mathrm{im}(P_A) = A$. This is because, if $b \in B$, then for any $a \in A$ we have $(a, b) \in A \times B$, and $P_A((a, b)) = a$.

If $B = \emptyset$, then $A \times B = \emptyset$, so P_A is the empty function, and therefore $\mathrm{im}(P_A) = \emptyset$.

(b) What is the coimage of this function? (Make sure to consider the possibility where $B = \emptyset$.)

Recall that $\operatorname{coim}(P_A)$ is the set of fibers of P_A. If $B \neq \emptyset$, then $\operatorname{im}(P_A) = A$, as we showed in Part (a). The fiber over any $a \in \operatorname{im}(P_A) = A$ is $f^{-1}(a) = \{a\} \times B$. Thus

$$\operatorname{coim}(P_A) = \{\{a\} \times B \mid a \in A\}.$$

If $B = \emptyset$, then $A \times B = \emptyset$, so no fibers are possible for P_A. Hence $\operatorname{coim}(P_A) = \emptyset$.