The Determinant

Definition

Definition

Definition.

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)

Definition

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$.

Definition

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D\colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

Definition

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D\colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

that is

- multilinear,

Definition

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D \colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

that is

- multilinear,

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D\colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

that is

- multilinear,
- alternating, and

Definition

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D\colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

that is

- multilinear,
- alternating, and

Definition

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D\colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

that is

- multilinear,
- alternating, and
- normalized so that $D(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

Definition

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D\colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

that is

- multilinear,
- alternating, and
- normalized so that $D(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

Definition

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D\colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

that is

- multilinear,
- alternating, and
- normalized so that $D(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

Since D accepts n column vectors of length n, one may view D as a function $D\colon M_{n\times n}(\mathbb{F}) \to \mathbb{F}$.

Definition. (Slightly different than Chap 4, Section 1.2, Defn 2.1.)
Let $\mathbb{V} = \mathbb{F}^n$. The **determinant** is an n-ary scalar-valued function

$$D\colon \mathbb{V} \times \cdots \times \mathbb{V} \to \mathbb{F}$$

that is

- multilinear,
- alternating, and
- normalized so that $D(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$.

Since D accepts n column vectors of length n, one may view D as a function $D\colon M_{n\times n}(\mathbb{F}) \to \mathbb{F}$.

For $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{V}$, may write $D(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ or $D\left(\begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix}\right)$ or $\det\left(\begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix}\right)$ or $\begin{vmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{vmatrix}$ for the determinant.

Multilinear functions are determined by behavior on a basis

Multilinear functions are determined by behavior on a basis

Example.

Multilinear functions are determined by behavior on a basis

Example. Consider a bilinear function $B\colon \mathbb{F}^3 \to \mathbb{W}$.

Multilinear functions are determined by behavior on a basis

Example. Consider a bilinear function $B\colon \mathbb{F}^3 \to \mathbb{W}$. We want to evaluate

$$B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right).$$

Multilinear functions are determined by behavior on a basis

Example. Consider a bilinear function $B\colon \mathbb{F}^3 \to \mathbb{W}$. We want to evaluate $B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right)$. Write $\begin{bmatrix} a \\ c \\ e \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\,\mathbf{e}_1 + c\,\mathbf{e}_2 + e\,\mathbf{e}_3$.

Multilinear functions are determined by behavior on a basis

Example. Consider a bilinear function $B\colon \mathbb{F}^3 \to \mathbb{W}$. We want to evaluate $B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right)$. Write $\begin{bmatrix} a \\ c \\ e \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\,\mathbf{e}_1 + c\,\mathbf{e}_2 + e\,\mathbf{e}_3$. Hence

$$B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) \quad = a\,B\left(\mathbf{e}_1, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) + c\,B\left(\mathbf{e}_2, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) + e\,B\left(\mathbf{e}_3, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right)$$

Example. Consider a bilinear function $B\colon \mathbb{F}^3 \to \mathbb{W}$. We want to evaluate $B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right)$. Write $\begin{bmatrix} a \\ c \\ e \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\,\mathbf{e}_1 + c\,\mathbf{e}_2 + e\,\mathbf{e}_3$. Hence

$$
\begin{aligned}
B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) &= a\,B\left(\mathbf{e}_1, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) + c\,B\left(\mathbf{e}_2, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) + e\,B\left(\mathbf{e}_3, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) \\
&= ab\,B(\mathbf{e}_1, \mathbf{e}_1) + ad\,B(\mathbf{e}_1, \mathbf{e}_2) + af\,B(\mathbf{e}_1, \mathbf{e}_3) + \\
&\quad + cb\,B(\mathbf{e}_2, \mathbf{e}_1) + cd\,B(\mathbf{e}_2, \mathbf{e}_2) + cf\,B(\mathbf{e}_2, \mathbf{e}_3) + \\
&\quad + eb\,B(\mathbf{e}_3, \mathbf{e}_1) + ed\,B(\mathbf{e}_3, \mathbf{e}_2) + ef\,B(\mathbf{e}_3, \mathbf{e}_3)\,.
\end{aligned}
$$

Multilinear functions are determined by behavior on a basis

Example. Consider a bilinear function $B\colon \mathbb{F}^3 \to \mathbb{W}$. We want to evaluate $B\left(\begin{bmatrix}a\\c\\e\end{bmatrix}, \begin{bmatrix}b\\d\\f\end{bmatrix}\right)$. Write $\begin{bmatrix}a\\c\\e\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + c\begin{bmatrix}0\\1\\0\end{bmatrix} + e\begin{bmatrix}0\\0\\1\end{bmatrix} = a\,\mathbf{e}_1 + c\,\mathbf{e}_2 + e\,\mathbf{e}_3$. Hence

$$\begin{aligned}
B\left(\begin{bmatrix}a\\c\\e\end{bmatrix}, \begin{bmatrix}b\\d\\f\end{bmatrix}\right) &= a\,B\left(\mathbf{e}_1, \begin{bmatrix}b\\d\\f\end{bmatrix}\right) + c\,B\left(\mathbf{e}_2, \begin{bmatrix}b\\d\\f\end{bmatrix}\right) + e\,B\left(\mathbf{e}_3, \begin{bmatrix}b\\d\\f\end{bmatrix}\right)\\
&= ab\,B(\mathbf{e}_1,\mathbf{e}_1) + ad\,B(\mathbf{e}_1,\mathbf{e}_2) + af\,B(\mathbf{e}_1,\mathbf{e}_3) +\\
&\quad + cb\,B(\mathbf{e}_2,\mathbf{e}_1) + cd\,B(\mathbf{e}_2,\mathbf{e}_2) + cf\,B(\mathbf{e}_2,\mathbf{e}_3) +\\
&\quad + eb\,B(\mathbf{e}_3,\mathbf{e}_1) + ed\,B(\mathbf{e}_3,\mathbf{e}_2) + ef\,B(\mathbf{e}_3,\mathbf{e}_3)\,.
\end{aligned}$$

Any B defined by the blue formula, with $B(\mathbf{e}_i, \mathbf{e}_j) \in \mathbb{W}$ chosen arbitrarily, will be bilinear.

Alternating multilinear functions are more determined

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.
Recall also that any alternating multilinear function is antisymmetric.

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.
Recall also that any alternating multilinear function is antisymmetric.

If $B(\mathbf{x}, \mathbf{y})$ is alternating and bilinear, we can simplify the previous Example, since (for example) $B(\mathbf{e}_1, \mathbf{e}_1) = 0$ and $B(\mathbf{e}_2, \mathbf{e}_1) = -B(\mathbf{e}_1, \mathbf{e}_2)$.

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.
Recall also that any alternating multilinear function is antisymmetric.

If $B(\mathbf{x}, \mathbf{y})$ is alternating and bilinear, we can simplify the previous Example, since (for example) $B(\mathbf{e}_1, \mathbf{e}_1) = 0$ and $B(\mathbf{e}_2, \mathbf{e}_1) = -B(\mathbf{e}_1, \mathbf{e}_2)$. This yields

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.
Recall also that any alternating multilinear function is antisymmetric.

If $B(\mathbf{x}, \mathbf{y})$ is alternating and bilinear, we can simplify the previous Example, since (for example) $B(\mathbf{e}_1, \mathbf{e}_1) = 0$ and $B(\mathbf{e}_2, \mathbf{e}_1) = -B(\mathbf{e}_1, \mathbf{e}_2)$. This yields

$$\begin{aligned} B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix} \right) \quad &= ab\, B\left(\mathbf{e}_1, \mathbf{e}_1\right) + ad\, B\left(\mathbf{e}_1, \mathbf{e}_2\right) + af\, B\left(\mathbf{e}_1, \mathbf{e}_3\right) + \\ &+ cb\, B\left(\mathbf{e}_2, \mathbf{e}_1\right) + cd\, B\left(\mathbf{e}_2, \mathbf{e}_2\right) + cf\, B\left(\mathbf{e}_2, \mathbf{e}_3\right) + \\ &+ eb\, B\left(\mathbf{e}_3, \mathbf{e}_1\right) + ed\, B\left(\mathbf{e}_3, \mathbf{e}_2\right) + ef\, B\left(\mathbf{e}_3, \mathbf{e}_3\right) \end{aligned}$$

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.
Recall also that any alternating multilinear function is antisymmetric.

If $B(\mathbf{x}, \mathbf{y})$ is alternating and bilinear, we can simplify the previous Example, since (for example) $B(\mathbf{e}_1, \mathbf{e}_1) = 0$ and $B(\mathbf{e}_2, \mathbf{e}_1) = -B(\mathbf{e}_1, \mathbf{e}_2)$. This yields

$$
\begin{aligned}
B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) &= ab\, B\left(\mathbf{e}_1, \mathbf{e}_1\right) + ad\, B\left(\mathbf{e}_1, \mathbf{e}_2\right) + af\, B\left(\mathbf{e}_1, \mathbf{e}_3\right) + \\
&\quad + cb\, B\left(\mathbf{e}_2, \mathbf{e}_1\right) + cd\, B\left(\mathbf{e}_2, \mathbf{e}_2\right) + cf\, B\left(\mathbf{e}_2, \mathbf{e}_3\right) + \\
&\quad + eb\, B\left(\mathbf{e}_3, \mathbf{e}_1\right) + ed\, B\left(\mathbf{e}_3, \mathbf{e}_2\right) + ef\, B\left(\mathbf{e}_3, \mathbf{e}_3\right) \\
&= (ad - cb) B\left(\mathbf{e}_1, \mathbf{e}_2\right) + (af - eb) B\left(\mathbf{e}_1, \mathbf{e}_3\right) + (cf - ed) B\left(\mathbf{e}_2, \mathbf{e}_3\right).
\end{aligned}
$$

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.
Recall also that any alternating multilinear function is antisymmetric.

If $B(\mathbf{x}, \mathbf{y})$ is alternating and bilinear, we can simplify the previous Example, since (for example) $B(\mathbf{e}_1, \mathbf{e}_1) = 0$ and $B(\mathbf{e}_2, \mathbf{e}_1) = -B(\mathbf{e}_1, \mathbf{e}_2)$. This yields

$$\begin{aligned} B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) &= ab\, B\left(\mathbf{e}_1, \mathbf{e}_1\right) + ad\, B\left(\mathbf{e}_1, \mathbf{e}_2\right) + af\, B\left(\mathbf{e}_1, \mathbf{e}_3\right) + \\ &+cb\, B\left(\mathbf{e}_2, \mathbf{e}_1\right) + cd\, B\left(\mathbf{e}_2, \mathbf{e}_2\right) + cf\, B\left(\mathbf{e}_2, \mathbf{e}_3\right) + \\ &+eb\, B\left(\mathbf{e}_3, \mathbf{e}_1\right) + ed\, B\left(\mathbf{e}_3, \mathbf{e}_2\right) + ef\, B\left(\mathbf{e}_3, \mathbf{e}_3\right) \\ &= (ad - cb)B\left(\mathbf{e}_1, \mathbf{e}_2\right) + (af - eb)B\left(\mathbf{e}_1, \mathbf{e}_3\right) + (cf - ed)B\left(\mathbf{e}_2, \mathbf{e}_3\right). \end{aligned}$$

We see that an alternating bilinear transformation is determined by its values $B(\mathbf{e}_i, \mathbf{e}_j)$ with $i < j$.

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.
Recall also that any alternating multilinear function is antisymmetric.

If $B(\mathbf{x},\mathbf{y})$ is alternating and bilinear, we can simplify the previous Example, since (for example) $B(\mathbf{e}_1,\mathbf{e}_1) = 0$ and $B(\mathbf{e}_2,\mathbf{e}_1) = -B(\mathbf{e}_1,\mathbf{e}_2)$. This yields

$$\begin{aligned} B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) &= ab\, B(\mathbf{e}_1,\mathbf{e}_1) + ad\, B(\mathbf{e}_1,\mathbf{e}_2) + af\, B(\mathbf{e}_1,\mathbf{e}_3) + \\ &\quad + cb\, B(\mathbf{e}_2,\mathbf{e}_1) + cd\, B(\mathbf{e}_2,\mathbf{e}_2) + cf\, B(\mathbf{e}_2,\mathbf{e}_3) + \\ &\quad + eb\, B(\mathbf{e}_3,\mathbf{e}_1) + ed\, B(\mathbf{e}_3,\mathbf{e}_2) + ef\, B(\mathbf{e}_3,\mathbf{e}_3) \\ &= (ad-cb)B(\mathbf{e}_1,\mathbf{e}_2) + (af-eb)B(\mathbf{e}_1,\mathbf{e}_3) + (cf-ed)B(\mathbf{e}_2,\mathbf{e}_3)\,. \end{aligned}$$

We see that an alternating bilinear transformation is determined by its values $B(\mathbf{e}_i,\mathbf{e}_j)$ with $i<j$. (Similarly, an alternating trilinear transformation is determined by its values $B(\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k)$ with $i<j<k$.)

Alternating multilinear functions are more determined

Recall that a multilinear function is **alternating** is it returns the value 0 whenever two inputs are equal.
Recall also that any alternating multilinear function is antisymmetric.

If $B(\mathbf{x}, \mathbf{y})$ is alternating and bilinear, we can simplify the previous Example, since (for example) $B(\mathbf{e}_1, \mathbf{e}_1) = 0$ and $B(\mathbf{e}_2, \mathbf{e}_1) = -B(\mathbf{e}_1, \mathbf{e}_2)$. This yields

$$\begin{aligned} B\left(\begin{bmatrix} a \\ c \\ e \end{bmatrix}, \begin{bmatrix} b \\ d \\ f \end{bmatrix}\right) &= ab\, B\left(\mathbf{e}_1, \mathbf{e}_1\right) + ad\, B\left(\mathbf{e}_1, \mathbf{e}_2\right) + af\, B\left(\mathbf{e}_1, \mathbf{e}_3\right) + \\ &\quad + cb\, B\left(\mathbf{e}_2, \mathbf{e}_1\right) + cd\, B\left(\mathbf{e}_2, \mathbf{e}_2\right) + cf\, B\left(\mathbf{e}_2, \mathbf{e}_3\right) + \\ &\quad + eb\, B\left(\mathbf{e}_3, \mathbf{e}_1\right) + ed\, B\left(\mathbf{e}_3, \mathbf{e}_2\right) + ef\, B\left(\mathbf{e}_3, \mathbf{e}_3\right) \\ &= (ad - cb) B\left(\mathbf{e}_1, \mathbf{e}_2\right) + (af - eb) B\left(\mathbf{e}_1, \mathbf{e}_3\right) + (cf - ed) B\left(\mathbf{e}_2, \mathbf{e}_3\right). \end{aligned}$$

We see that an alternating bilinear transformation is determined by its values $B(\mathbf{e}_i, \mathbf{e}_j)$

with $i < j$. (Similarly, an alternating trilinear transformation is determined by its values $B(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$ with $i < j < k$.)

Uniqueness of the determinant

Uniqueness of the determinant

It is not hard to see that an alternating, n-linear transformation on n-dimensional space is determined by $B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.

Uniqueness of the determinant

It is not hard to see that an alternating, n-linear transformation on n-dimensional space is determined by $B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.
This is the reason that there is a unique alternating n-linear scalar-valued function $D\colon \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ that is normalized by setting $D(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = 1$.

Uniqueness of the determinant

It is not hard to see that an alternating, n-linear transformation on n-dimensional space is determined by $B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.
This is the reason that there is a unique alternating n-linear scalar-valued function $D\colon \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ that is normalized by setting $D(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = 1$.

Example.

Uniqueness of the determinant

It is not hard to see that an alternating, n-linear transformation on n-dimensional space is determined by $B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.
This is the reason that there is a unique alternating n-linear scalar-valued function $D\colon \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ that is normalized by setting $D(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = 1$.

Example. (The determinant on $\mathbb{F}^2$)

Uniqueness of the determinant

It is not hard to see that an alternating, n-linear transformation on n-dimensional space is determined by $B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.
This is the reason that there is a unique alternating n-linear scalar-valued function $D\colon \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ that is normalized by setting $D(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = 1$.

Example. (The determinant on $\mathbb{F}^2$)

$$
\begin{aligned}
D\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) &= ab\, D(\mathbf{e}_1, \mathbf{e}_1) + ad\, D(\mathbf{e}_1, \mathbf{e}_2) + \\
&\quad + cb\, D(\mathbf{e}_2, \mathbf{e}_1) + cd\, D(\mathbf{e}_2, \mathbf{e}_2)
\end{aligned}
$$

Uniqueness of the determinant

It is not hard to see that an alternating, n-linear transformation on n-dimensional space is determined by $B(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$.
This is the reason that there is a unique alternating n-linear scalar-valued function $D\colon \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ that is normalized by setting $D(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n) = 1$.

Example. (The determinant on $\mathbb{F}^2$)

$$
\begin{aligned}
D\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) &= ab\, D(\mathbf{e}_1, \mathbf{e}_1) + ad\, D(\mathbf{e}_1, \mathbf{e}_2) + \\
&\quad + cb\, D(\mathbf{e}_2, \mathbf{e}_1) + cd\, D(\mathbf{e}_2, \mathbf{e}_2) \\
&= (ad - cb) D(\mathbf{e}_1, \mathbf{e}_2) \\
&= ad - cb.
\end{aligned}
$$

Uniqueness of the determinant

It is not hard to see that an alternating, n-linear transformation on n-dimensional space is determined by $B(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$.
This is the reason that there is a unique alternating n-linear scalar-valued function $D\colon \mathbb{F}^n \times \cdots \times \mathbb{F}^n \to \mathbb{F}$ that is normalized by setting $D(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) = 1$.

Example. (The determinant on $\mathbb{F}^2$)

$$\begin{aligned} D\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\right) &= ab\, D\left(\mathbf{e}_1, \mathbf{e}_1\right) + ad\, D\left(\mathbf{e}_1, \mathbf{e}_2\right) + \\ &+ cb\, D\left(\mathbf{e}_2, \mathbf{e}_1\right) + cd\, D\left(\mathbf{e}_2, \mathbf{e}_2\right) \\ \\ &= (ad - cb) D\left(\mathbf{e}_1, \mathbf{e}_2\right) \\ &= ad - cb. \end{aligned}$$

We also write $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$

The 3×3-determinant

Example.

The 3×3-determinant

Example. (The determinant on $\mathbb{F}^3$)

Example. (The determinant on $\mathbb{F}^3$)

$$D\left(\begin{bmatrix} a \\ d \\ g \end{bmatrix}, \begin{bmatrix} b \\ e \\ h \end{bmatrix}, \begin{bmatrix} c \\ f \\ i \end{bmatrix}\right) = \text{(a sum of 27 terms)}$$

$$\begin{aligned} &= (aei + bfg + cdh - afh - bdi - ceg) D(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \\ &= aei + bfg + cdh - afh - bdi - ceg. \end{aligned}$$

The permutation expansion of the determinant, I

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \ldots, n\}$ is a bijection $\pi\colon X \to X$ from X to itself.

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \ldots, n\}$ is a bijection $\pi \colon X \to X$ from X to itself.

A permutation that moves exactly two elements, and swaps those two, is called a **transposition**.

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \ldots, n\}$ is a bijection $\pi \colon X \to X$ from X to itself.

A permutation that moves exactly two elements, and swaps those two, is called a **transposition**.

Associated to a permutation, like $\pi \colon \{1, 2, 3\} \to \{1, 2, 3\} \colon 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$, is the corresponding **permutation matrix**:

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \ldots, n\}$ is a bijection $\pi\colon X \to X$ from X to itself.

A permutation that moves exactly two elements, and swaps those two, is called a **transposition**.

Associated to a permutation, like $\pi\colon \{1,2,3\} \to \{1,2,3\}\colon 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$, is the corresponding **permutation matrix**:

$$P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \dots, n\}$ is a bijection $\pi\colon X \to X$ from X to itself.

A permutation that moves exactly two elements, and swaps those two, is called a **transposition**.

Associated to a permutation, like $\pi\colon \{1,2,3\} \to \{1,2,3\}\colon 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$, is the corresponding **permutation matrix**:

$$P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is the matrix P having the property $P \cdot \mathbf{e}_i = \mathbf{e}_{\pi(i)}$.

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \ldots, n\}$ is a bijection $\pi\colon X \to X$ from X to itself.

A permutation that moves exactly two elements, and swaps those two, is called a **transposition**.

Associated to a permutation, like $\pi\colon \{1,2,3\} \to \{1,2,3\}\colon 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$, is the corresponding **permutation matrix**:

$$P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is the matrix P having the property $P \cdot \mathbf{e}_i = \mathbf{e}_{\pi(i)}$. It permutes the standard basis vectors the same way that π permutes X.

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \ldots, n\}$ is a bijection $\pi \colon X \to X$ from X to itself.

A permutation that moves exactly two elements, and swaps those two, is called a **transposition**.

Associated to a permutation, like $\pi \colon \{1, 2, 3\} \to \{1, 2, 3\} \colon 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$, is the corresponding **permutation matrix**:

$$P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is the matrix P having the property $P \cdot \mathbf{e}_i = \mathbf{e}_{\pi(i)}$. It permutes the standard basis vectors the same way that π permutes X. Hence the columns of P are $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in a permuted order.

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \ldots, n\}$ is a bijection $\pi \colon X \to X$ from X to itself.

A permutation that moves exactly two elements, and swaps those two, is called a **transposition**.

Associated to a permutation, like $\pi \colon \{1,2,3\} \to \{1,2,3\} \colon 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$, is the corresponding **permutation matrix**:

$$P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is the matrix P having the property $P \cdot \mathbf{e}_i = \mathbf{e}_{\pi(i)}$. It permutes the standard basis vectors the same way that π permutes X. Hence the columns of P are $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in a permuted order. This implies that P has exactly one 1 in each row and column, and the rest of the entries of P are 0.

The permutation expansion of the determinant, I

A **permutation** of $X = \{1, 2, \ldots, n\}$ is a bijection $\pi\colon X \to X$ from X to itself.

A permutation that moves exactly two elements, and swaps those two, is called a **transposition**.

Associated to a permutation, like $\pi\colon \{1,2,3\} \to \{1,2,3\}\colon 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$, is the corresponding **permutation matrix**:

$$P_\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This is the matrix P having the property $P \cdot \mathbf{e}_i = \mathbf{e}_{\pi(i)}$. It permutes the standard basis vectors the same way that π permutes X. Hence the columns of P are $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in a permuted order. This implies that P has exactly one 1 in each row and column, and the rest of the entries of P are 0. (This property characterizes permutation matrices.)

$$\det \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = (-1) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = (-1)^2 \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (-1)^2.$$

The permutation expansion of the determinant, II

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$.

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

It follows from the Fact that if a permutation π is the composition of k transpositions, then P_π can be transformed into the identity matrix with k column swaps.

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

It follows from the Fact that if a permutation π is the composition of k transpositions, then P_π can be transformed into the identity matrix with k column swaps. (And vice versa!)

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

It follows from the Fact that if a permutation π is the composition of k transpositions, then P_π can be transformed into the identity matrix with k column swaps. (And vice versa!)

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

It follows from the Fact that if a permutation π is the composition of k transpositions, then P_π can be transformed into the identity matrix with k column swaps. (And vice versa!)

We therefore call a permutation matrix **even** if it can be transformed into the identity matrix with an even number of column swaps, and **odd** if it can be transformed into the identity matrix with an odd number of column swaps.

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

It follows from the Fact that if a permutation π is the composition of k transpositions, then P_π can be transformed into the identity matrix with k column swaps. (And vice versa!)

We therefore call a permutation matrix **even** if it can be transformed into the identity matrix with an even number of column swaps, and **odd** if it can be transformed into the identity matrix with an odd number of column swaps. With this definition, $\det P_\pi = \mathrm{sign}(\pi)$.

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

It follows from the Fact that if a permutation π is the composition of k transpositions, then P_π can be transformed into the identity matrix with k column swaps. (And vice versa!)

We therefore call a permutation matrix **even** if it can be transformed into the identity matrix with an even number of column swaps, and **odd** if it can be transformed into the identity matrix with an odd number of column swaps. With this definition, $\det P_\pi = \operatorname{sign}(\pi)$.

[Explain how to compute the Cauchy number.]

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

It follows from the Fact that if a permutation π is the composition of k transpositions, then P_π can be transformed into the identity matrix with k column swaps. (And vice versa!)

We therefore call a permutation matrix **even** if it can be transformed into the identity matrix with an even number of column swaps, and **odd** if it can be transformed into the identity matrix with an odd number of column swaps. With this definition, $\det P_\pi = \operatorname{sign}(\pi)$.

[Explain how to compute the Cauchy number.]

Fact: $\det P_\pi = (-1)^{\text{Cauchy number}(\pi)}$.

The permutation expansion of the determinant, II

If a permutation π is the composition of k transpositions, then the **sign** of π is $(-1)^k$. If the sign is $+1$, then π is called **even**, and if the sign is -1, then π is called **odd**.

Fact: If $\pi = \tau_1 \circ \cdots \circ \tau_k$, each τ_i a transposition, then $P_\pi = P_{\tau_1} \cdots P_{\tau_k}$.

It follows from the Fact that if a permutation π is the composition of k transpositions, then P_π can be transformed into the identity matrix with k column swaps. (And vice versa!)

We therefore call a permutation matrix **even** if it can be transformed into the identity matrix with an even number of column swaps, and **odd** if it can be transformed into the identity matrix with an odd number of column swaps. With this definition, $\det P_\pi = \operatorname{sign}(\pi)$.

[Explain how to compute the Cauchy number.]

Fact: $\det P_\pi = (-1)^{\text{Cauchy number}(\pi)}$.

Permutation expansion. If $A = [a_{ij}]$, then

$$\det A = \sum_\pi \operatorname{sign}(\pi) a_{\pi(1)1} \cdots a_{\pi(n)n}.$$