Counting basics.

Definition 1. Let A and B be sets.

(i) If there exists a bijection from A to B, then we write $|A| = |B|$ and say *A and B have the same cardinality* (or A and B are *equinumerous*, or are *equipotent*).

(ii) If there exists an injection from A to B, then we write $|A| \leq |B|$ and say *the cardinality of A is less than or equal to the cardinality of B.*

(iii) $|A| < |B|$ means "$|A| \leq |B|$ and $|A| \neq |B|$".

(iv) If $n \in \mathbb{N}$, then we write $|A| = n$ to mean $|A| = |n|$, and we say that *A has cardinality n.*

(v) A is *finite* if $|A| = n$ for some natural number n. Otherwise A is *infinite.*

(vi) An infinite set A is *countable* if $|A| = |\mathbb{N}|$. Otherwise A is *uncountable.*

Comments.

(i) The relation of "having the same cardinality" is reflexive, symmetric and transitive. This is easy to show.

(ii) The Cantor-Schroeder-Bernstein Theorem asserts that

$$((|A| \leq |B|) \wedge (|B| \leq |A|)) \rightarrow |A| = |B|.$$

This is not easy to show. To prove it you must show that if there are injections $f\colon A \rightarrow B$ and $g\colon B \rightarrow A$, then there is a bijection $h\colon A \rightarrow B$. You can read a proof at `http://en.wikipedia.org/wiki/Cantor-Bernstein-Schroeder_theorem`

(iii) Cantor's Theorem asserts that $|A| < |\mathcal{P}(A)|$.

(iv) Certain basic facts require proof, which we will not give in class:

(a) If $m, n \in \mathbb{N}$ and $|m| = |n|$, then $m = n$.

(b) $\mathbb{N}$ is infinite.

(c) A subset S of a finite set F is itself finite, and $|S| \leq |F|$. If S is a proper subset, then $|S| < |F|$.

(d) The union of two finite sets is finite.

Two basic theorems about counting.

(1) The Sum Rule: If A and B are disjoint finite sets, then $|A \cup B| = |A| + |B|$. (This means: if $A \cap B = \emptyset$, $|A| = m$ and $|B| = n$, then $|A \cup B| = m + n$.)

(2) The Product Rule: If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$. (This means what you think.)

Frequently the need to use the Sum Rule is signified by (exclusive) **or**, while the need to use the Product Rule is signified by (independent) **and**. For example, suppose you allow yourself to choose a dessert that is either an ice cream cone **or** a sundae, but not both. Suppose the type of ice cream cone is determined by one of 31 flavors of ice cream **and** one of 3 types of cone, while the choice of sundae is determined by selecting one of 7 possible toppings **and** one of 4 possible sizes. How many dessert choices do you have? Answer: $(31 \times 3) + (7 \times 4) = 93 + 28 = 121$ choices.

Exercises.

(1) Show that the empty set is finite.
(2) Show that $\mathbb{N}$ is countable.
(3) Show that $\mathbb{Z}$ is countable.
(4) How many license plates have exactly 7 characters consisting of decimal digits (0,1,2,. . . ,9) and letters of the alphabet (a, b, c, . . . , z)? What if there are exactly 3 decimal digits, 4 letters of the alphabet, and the digits must come before the letters?
(5) How many license plates have 3 decimal digits and 4 letters of the alphabet if the plate must start and end with a digit, and either start with 9 or end with 9?
(6) How many 0,1-sequences of length n are there?
(7) Show that if $|A| = n$, then $|\mathcal{P}(A)| = 2^n$.