Classification of $\mathbb{F}$-vector spaces.

The main point of these notes is to establish that every $\mathbb{F}$-vector space is determined up to isomorphism by its dimension. That is, every $\mathbb{F}$-vector space has a dimension, and any two $\mathbb{F}$-vector spaces of the same dimension are isomorphic.

Theorem 1. *Every vector space has a basis.*

Idea of proof. Enumerate a spanning set $S \subseteq \mathbb{V}$ using an ordinal number. (If you are not given a spanning set, you can always take $S = \mathbb{V}$, since $\mathbb{V}$ spans itself.) We have

$$v_0, v_1, v_2, \ldots, v_\omega, v_{\omega+1}, v_{\omega+2}, \ldots.$$

Using the algorithm described on page 5 of the slides "basis.pdf" from February 22, examine the vectors in this list one at a time and delete those that depend on the set of vectors listed earlier. When the process is complete, the set of undeleted vectors will be a basis. □

Theorem 2. *If $\mathbb{V}$ and $\mathbb{W}$ are $\mathbb{F}$-spaces, then $\mathbb{V} \cong \mathbb{W}$ if and only if there exist bases $\mathcal{B}$ and $\mathcal{C}$ for $\mathbb{V}$ and $\mathbb{W}$ respectively and a bijection $f \colon \mathcal{B} \to \mathcal{C}$.*

The proof of this theorem uses the Universal Mapping Property, which is proved later.

Proof. Assume that $f \colon \mathcal{B} \to \mathcal{C}$ is a bijection. By the Universal Mapping Property, there exists a unique linear transformation $\overline{f} \colon \mathbb{V} \to \mathbb{W}$ extending f. Similarly, there exists a unique linear transformation $\overline{f^{-1}} \colon \mathbb{W} \to \mathbb{V}$ extending $f^{-1} \colon \mathcal{C} \to \mathcal{B}$. The compositions $\overline{f^{-1}} \circ \overline{f}$ and $\overline{f} \circ \overline{f^{-1}}$ extend the identity functions $f^{-1} \circ f = \mathrm{id}_{\mathcal{B}}$ and $f \circ f^{-1} = \mathrm{id}_{\mathcal{C}}$, so, by the uniqueness of extensions, $\overline{f^{-1}} \circ \overline{f} = \mathrm{id}_{\mathbb{V}}$ and $\overline{f} \circ \overline{f^{-1}} = \mathrm{id}_{\mathbb{W}}$. This proves that $\overline{f}$ and $\overline{f^{-1}}$ are inverse isomorphisms between $\mathbb{V}$ and $\mathbb{W}$.

Conversely, assume that $g \colon \mathbb{V} \to \mathbb{W}$ is an isomorphism. If $\mathcal{B}$ is a basis for $\mathbb{V}$, then I claim that $\mathcal{C} := g(\mathcal{B})$ is a basis for $\mathbb{W}$ and g restricts to be a bijection from $\mathcal{B}$ to $\mathcal{C}$. To justify this we need to show three things:

(1) The restriction $g|_{\mathcal{B}}$ is a bijection from $\mathcal{B}$ to $\mathcal{C}$.

It suffices to observe that $g^{-1}|_{\mathcal{C}}$ is the inverse of $g|_{\mathcal{B}}$. That is, $g^{-1}|_{\mathcal{C}} \circ g|_{\mathcal{B}}(b) = b$ for any $b \in \mathcal{B}$ and $g|_{\mathcal{B}} \circ g^{-1}|_{\mathcal{C}}(c) = c$ for any $c \in \mathcal{C}$.

(2) $\mathcal{C}$ spans $\mathcal{W}$.

We argue that an arbitrarily chosen vector $w \in \mathbb{W}$ belongs to $\mathrm{span}(\mathcal{C})$. Write $v := g^{-1}(w)$ in terms of the basis $\mathcal{B}$ as $v = \sum \alpha_i b_i$, and then apply g to this equality: $w = g(v) = \sum \alpha_i g(b_i) \in \mathrm{span}(\mathcal{C})$.

(3) $\mathcal{C}$ is linearly independent.

If $\sum \alpha_i c_i = 0$, then $\sum \alpha_i g^{-1}(c_i) = g^{-1}(\sum \alpha_i c_i) = 0$, since g is 1-1, so $\alpha_i = 0$ for all i, since $g^{-1}(c_i) \in \mathcal{B}$ and $\mathcal{B}$ is a basis.

□

Next, let's state and prove the Universal Mapping Property, which guarantees that every linear transformation is determined by how it behaves on a basis.

Theorem 3. *(Universal Mapping Property) Let $\mathbb{V}, \mathbb{W}$ be $\mathbb{F}$-vector spaces. Let $\mathcal{B}$ be a basis for $\mathbb{V}$, If $f\colon \mathcal{B} \to \mathbb{W}$ is any function, then f extends uniquely to a linear transformation $\overline{f}\colon \mathbb{V} \to \mathbb{W}$.*

Here, when we say that $\overline{f}$ "extends" f, we mean that the two functions agree wherever f is defined (i.e. $\overline{f}(x) = f(x)$ whenever $x \in \text{domain}(f) = \mathcal{B}$). When we say that $\overline{f}$ is the "unique" extension of f to a linear transformation, we mean that $\overline{f}\colon \mathbb{V} \to \mathbb{W}$ is a linear transformation extending f and if $T\colon \mathbb{V} \to \mathbb{W}$ is any linear transformation extending f, then $T = \overline{f}$ (i.e. $T(x) = \overline{f}(x)$ for all $x \in \text{domain}(\overline{f}) = \mathbb{V}$).

Proof. Given $f\colon \mathcal{B} \to \mathbb{W}$, there is at most one way to extend f to the linear combinations of elements of $\mathcal{B}$ if the extension map must respect addition and scaling, and that is

$$\overline{f}\left(\sum \alpha_i b_i\right) := \sum \overline{f}(\alpha_i b_i) = \sum \alpha_i \overline{f}(b_i) = \sum \alpha_i f(b_i).$$

That is, if $v = \sum \alpha_i b_i \in \text{span}(\mathcal{B}) = \mathbb{V}$, then $\overline{f}(v)$ is defined to equal $\sum \alpha_i f(b_i)$.

There are things we must check, namely

(1) $\overline{f}$ is a function from $\mathbb{V}$ to $\mathbb{W}$.
(2) $\overline{f}$ extends f.
(3) $\overline{f}$ is linear.
(4) $\overline{f}$ is the unique linear transformation $\mathbb{V} \to \mathbb{W}$ that is an extension f.

Let's check them.

For Item (1), given any $v \in \mathbb{V}$ there is a unique way to write v in the $\mathcal{B}$-basis, say $v = \sum \alpha_i b_i \in \text{span}(\mathcal{B}) = \mathbb{V}$. Given this expression, we have defined $\overline{f}(v)$ to be $\sum \alpha_i f(b_i) \in \mathbb{W}$. That is, for each $v \in \mathbb{V}$, we have defined $\overline{f}(v)$ uniquely, so $\overline{f}$ is a function.

For Item (2), if $v = b \in \mathcal{B}$, then the unique expression of v as a linear combination of elements of $\mathcal{B}$ is $v = b$. We have defined $\overline{f}$ so that $\overline{f}(v) = \overline{f}(b) = f(b)$ in this case.

For Item (3), suppose that $u, v \in \mathbb{V}$ and $u = \sum \beta_j b_j$, $v = \sum \alpha_i b_i$. Then $u + v = \sum \beta_j b_j + \sum \alpha_i b_i$ so $\overline{f}(u + v) = \sum \beta_j f(b_j) + \sum \alpha_i f(b_i) = \overline{f}(u) + \overline{f}(v)$. This proves that $\overline{f}$ respects addition. If $r \in \mathbb{F}$, then $rv = r(\sum \alpha_i b_i) = \sum r\alpha_i b_i$. Therefore $\overline{f}(ru) = \sum r\alpha_i f(b_i) = r \sum \alpha_i f(b_i) = r\overline{f}(u)$. This proves that $\overline{f}$ respects scaling. Thus, $\overline{f}$ is linear.

For Item (4), assume that T is any linear transformation from $\mathbb{V}$ to $\mathbb{W}$ that extends f. Then, for $v = \sum \alpha_i b_i$ we must have $T(v) = T\left(\sum \alpha_i b_i\right) = \sum \alpha_i T(b_i) = \sum \alpha_i f(b_i) = \overline{f}(v)$, so T agrees with $\overline{f}$ at each $v \in \mathbb{V}$. This forces $T = \overline{f}$. $\square$