Numbers beyond $\mathbb{N}$

$$0, 1, 2, \ldots, \omega = \aleph_0, \omega + 1, \omega + 2, \ldots, \aleph_1, \ldots$$

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Examples. Any $n \in \omega$ is finite.

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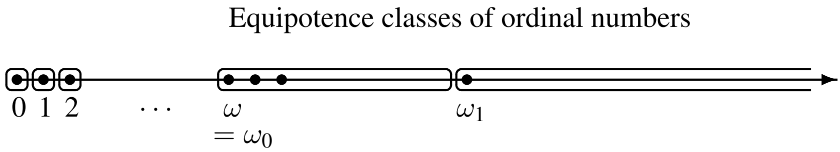

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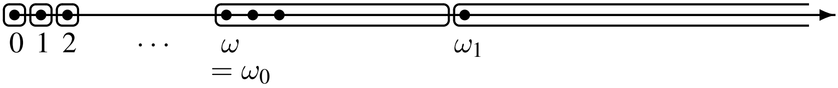

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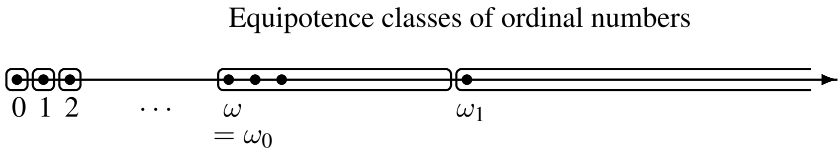

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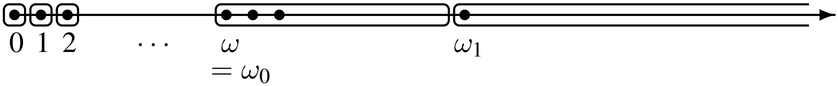

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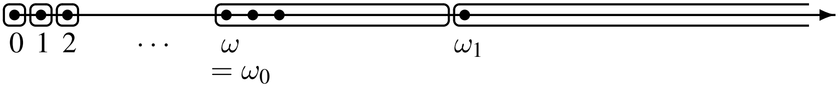

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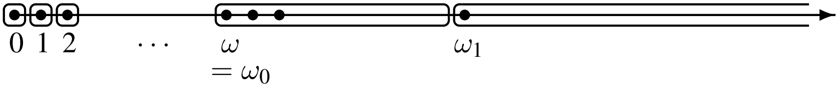

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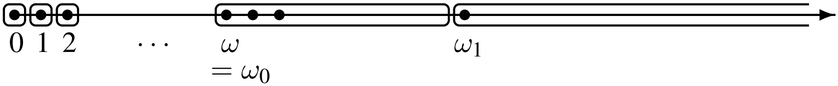

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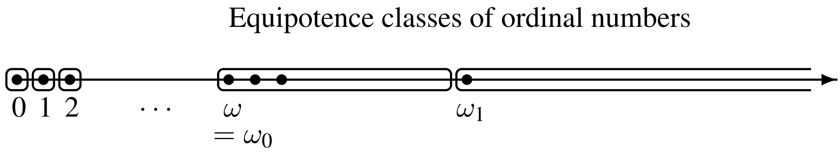

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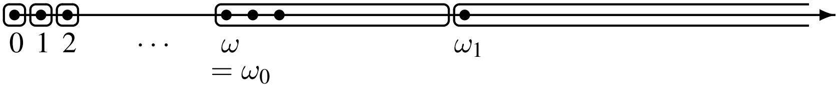

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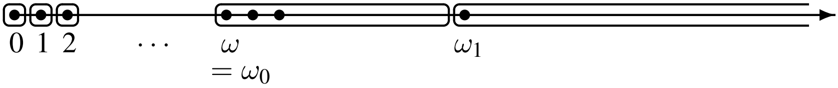

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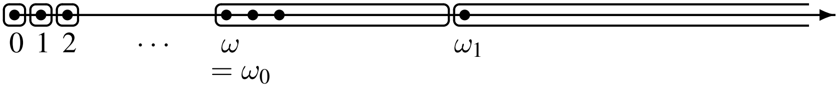

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