# Some uncountable sets

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Idea of proof: Try to build bijections (or almost-bijections) between subsets of $\mathbb{N}$, characteristic functions of subsets of $\mathbb{N}$, and base 2 expansions of real numbers from $(0,1)$.

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**Thm.** (ZFC) Let $A$ and $B$ be nonempty sets with at least one of them infinite.

$$|A \cup B| = |A \times B| = \max(|A|, |B|).$$