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(The same is true even if $\mathbb{V}$ is not finitely generated, but the proof in the non-finitely generated case requires some background from set theory.)

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3. If H depends on G and G depends on F, then H depends on F.

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Proof. Enumerate G as $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Recursively construct sets I_k by

1. $I_0 = \emptyset$, and
2. $I_{k+1} = I_k \cup \{\mathbf{v}_{k+1}\}$ if $\mathbf{v}_{k+1}$ is independent of I_k, while $I_{k+1} = I_k$ if $\mathbf{v}_{k+1}$ depends on I_k.

To verify:

1. Each I_k is independent.
2. $\operatorname{span}(I_k) = \operatorname{span}(\{\mathbf{v}_1, \ldots, \mathbf{v}_k\})$.
3. $I := I_n$ is an independent subset of G that spans the same subspace as G. $\square$

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Corollary. If $\mathbb{V}$ is finitely generated, then $\mathbb{V}$ has a finite basis and all bases have the same cardinality, say d. All independent subsets of $\mathbb{V}$ have size at most d, and all spanning subsets of $\mathbb{V}$ have size at least d.